A PRECISE RESULT ON THE ARITHMETIC OF NON-PRINCIPAL ORDERS IN ALGEBRAIC NUMBER FIELDS

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ABSTRACT. Let R be an order in an algebraic number field. If R is a principal order, then many explicit results on its arithmetic are available. Among others, R is half-factorial if and only if the class group of R has at most two elements. Much less is known for non-principal orders. Using a new semigroup theoretical approach, we study half-factoriality and further arithmetical properties for non-principal orders in algebraic number fields.

1. Introduction and Main Result

Let R be a noetherian domain. Then every non-zero non-unit $a \in R$ can be written as a finite product of atoms, say $a = u_1 \cdot \ldots \cdot u_k$. In general, a has many essentially different factorizations into atoms. The non-uniqueness of factorizations of elements in R is measured by arithmetical invariants. For convenience, we briefly recall the definition of two classical invariants, the elasticity and the set of distances (details will be given in Section 2). In a factorization of an element $a \in R$ as above, the number of factors k is called the length of the factorization. Then the elasticity $\rho(a) \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ is defined as the supremum over all k/l where k and l are lengths of factorizations of a. Suppose that $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$, where $k < l$ and all u_i and all v_j are atoms of R. If a has no factorizations of length m with $k < m < l$, then $l - k$ is said to be a distance of two (successive) factorization lengths, and $\Delta(a) \subset \mathbb{N}$ is the set of all such distances. The elasticity $\rho(R)$ is the supremum over all $\rho(a)$, and the set of distances $\Delta(R)$ is the union of all $\Delta(a)$. Then $\rho(R) = 1$ if and only if $\Delta(R) = \emptyset$, and in this case R is said to be half-factorial.

In the last decade, abstract finiteness results for arithmetical invariants have been derived for large classes of noetherian domains (see [12, Theorem 2.11.9], or [16, 17] for recent progress). If the noetherian domain is integrally closed, then it is a Krull domain, and if in addition every divisor class contains a prime divisor, then methods from additive and combinatorial number theory allow one to obtain precise results on the arithmetic (see [13] for the role of combinatorial number theory in this context). By a precise result, we mean an explicit formula, say for the elasticity, in terms of the group invariants of the class group, or an explicit characterization of the extremal cases, say $\rho(R) = 1$, which asks, in other words, for an explicit characterization of half-factoriality.

Half-factoriality has been a central topic ever since the beginning of factorization theory (see the surveys [7, 10, 23], and [8, 9, 11, 19] for some recent results). A classical result due to Carlitz states that a ring of integers is half-factorial if and only if its class group has at most two elements (see [4]; there are analogous results for Krull monoids, but for simplicity we restrict our discussion here to rings of integers). If R is a ring of integers in an algebraic number field, then, for almost all elements $a \in R$, we have $\Delta(a) = \{1\}$, and hence their sets of lengths are arithmetical progressions with difference 1 (see [12, Theorem 9.4.11]). Precise results of such a type for non-principal orders are extremely rare. In contrast to the above density result for principal orders, it is even open whether a non-principal order contains a single element a with $1 \in \Delta(a)$. In 1984, F. Halter-Koch gave a characterization of half-factoriality for non-principal orders in quadratic number fields (see [12, Theorem 3.7.15], or [14]), but the general case remained wide open ([18, 22]).

The present paper is devoted to non-principal orders in algebraic number fields and studies half-factoriality and the question whether 1 occurs in the set of distances. Here is our main result.

Theorem 1.1. *Let $\mathcal{O}$ be a non-principal, locally half-factorial order in an algebraic number field and set $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \overline{\mathcal{O}})\}$.*

1. *If $|\operatorname{Pic}(\mathcal{O})| = 1$, then $\mathcal{O}$ is half-factorial.*

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2. *If* $|\operatorname{Pic}(\mathcal{O})| \geq 3$, *then* $(\mathsf{D}(\operatorname{Pic}(\mathcal{O})))^2 \geq \mathsf{c}(\mathcal{O}) \geq 3$, $\min \triangle(\mathcal{O}) = 1$, *and* $\rho(\mathcal{O}) > 1$.
3. *If* $|\operatorname{Pic}(\mathcal{O})| = 2$, *then* $\rho(\mathcal{O}) \leq 2$, $2 \leq \mathsf{c}(\mathcal{O}) \leq 4$, *and* $\min \triangle(\mathcal{O}) \leq 1$.
   If, additionally, all localizations of $\mathcal{O}$ *are finitely primary monoids of exponent* 1*, then, setting* $k = \#\{\mathfrak{p} \in \mathcal{P}^* \mid [\overline{\mathcal{O}}_{\mathfrak{p}}^{\times}/\mathcal{O}_{\mathfrak{p}}^{\times}]_{\operatorname{Pic}(\mathcal{O})} = \operatorname{Pic}(\mathcal{O})\}$*, it follows that*
   - $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = \mathsf{c}(\mathcal{O}) = 2 + \min\{2, k\} \in \{2, 3, 4\}$;
   - $\rho(\mathcal{O}) = \frac{1}{2}\mathsf{c}(\mathcal{O}) \in \{1, \frac{3}{2}, 2\}$;
   - $\triangle(\mathcal{O}) = [1, \mathsf{c}(\mathcal{O}) - 2] \subset [1, 2]$;

   and the following are equivalent:
   - $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = 2$.
   - $\mathsf{c}(\mathcal{O}) = 2$.
   - $\mathcal{O}$ *is half-factorial.*

   If, additionally, $[\mathfrak{p}] = \mathbf{0}_{\operatorname{Pic}(\mathcal{O})}$ *for all* $\mathfrak{p} \in \mathcal{P}^*$*, then the following is also equivalent:*
   - $\mathsf{t}(\mathcal{O}) = 2$.

In particular, $\min \triangle(\mathcal{O}) \leq 1$ *always holds.*

Recall that $\mathcal{O}$ is called locally half-factorial if the localizations $\mathcal{O}_{\mathfrak{p}}$ are half-factorial for all non-zero prime ideals $\mathfrak{p}$ of $\mathcal{O}$. It is the standing conjecture that all half-factorial orders are locally half-factorial, and this holds true for orders in quadratic and cubic number fields. In particular, the above theorem yields the classical result of F. Halter-Koch as a corollary (see Corollary 4.7). We will see that the most difficult case is $|\operatorname{Pic}(\mathcal{O})| = 2$, and that the other ones are quite easy.

We briefly sketch our approach. We proceed in two steps. The first one is fairly standard in this area. We consider $\mathcal{O}$, the set of invertible ideals $\mathcal{I}^*(\mathcal{O})$, and construct the associated T-block monoid $\mathcal{B}(G, T, \iota)$. Then all questions under consideration can be studied in the T-block monoid instead of in $\mathcal{O}$ (see Section 3 for this transfer process). The second step contains the main new idea behind the present progress. In a series of recent papers (see for example [3, 5, 6]), arithmetical invariants of a monoid have been characterized in abstract semigroup theoretical terms, such as the monoid of relations and presentations. Of course, these semigroup theoretical invariants are far beyond reach in the case of non-principal orders. However, the T-block monoid $\mathcal{B}(G, T, \iota)$ has such simple constituents that these characterizations can be used to determine the arithmetical invariants exactly. These local results can be put together to get information for the whole T-block monoid $\mathcal{B}(G, T, \iota)$, and then all this is shifted to $\mathcal{O}$. Our crucial technical results are formulated in Lemma 3.16 and Proposition 3.17, which are based on [20] and [21].

In Section 2, we recall the relevant concepts from factorization theory and some abstract concepts from semigroup theory. In Section 3, we introduce T-block monoids and the associated transfer homomorphisms. The main work is to prove the already mentioned technical results Lemma 3.16 and Proposition 3.17. The proof of our main result, Theorem 1.1, will be given at the end of Section 3.

2. Preliminaries

In this note, our notation and terminology will be consistent with [12]. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \uplus \{0\}$. For integers n, $m \in \mathbb{Z}$, we set $[n, m] = \{x \in \mathbb{Z} \mid n \leq x \leq m\}$. By convention, the supremum of the empty set is zero and we set $\frac{0}{0} = 1$. The term "monoid" always means a commutative, cancellative semigroup with unit element. We will write all monoids multiplicatively. For a monoid H, we denote by $H^\times$ the set of invertible elements of H. We call H reduced if $H^\times = \{1\}$ and call $H_{\mathrm{red}} = H/H^\times$ the reduced monoid associated with H. Of course, H_{red} is always reduced. Note that the arithmetic of H is determined by H_{red}, and therefore we can restrict to reduced monoids whenever convenient. We denote by $\mathcal{A}(H)$ the *set of atoms* of H, by $\mathcal{A}(H_{\mathrm{red}})$ the set of atoms of the associated reduced monoid H_{red}, by $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ the free (abelian) monoid with basis $\mathcal{A}(H_{\mathrm{red}})$, and by $\pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the unique homomorphism such that $\pi_H | \mathcal{A}(H_{\mathrm{red}}) = \mathrm{id}$. We call $\mathsf{Z}(H)$ the *factorization monoid* and π_H the *factorization homomorphism* of H. For $a \in H$, we denote by $\mathsf{Z}(a) = \pi_H^{-1}(aH^\times)$ the *set of factorizations* of a and denote by $\mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\}$ the *set of lengths* of a, where $|\cdot|$ is the ordinary length function in the free monoid $\mathsf{Z}(H)$. In this terminology, a monoid H is called *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H \setminus H^\times$—this coincides with the classical definition of being half-factorial, since then every two factorizations of an element have the same length—and *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H \setminus H^\times$.

With all these notions at hand, for $a \in H$, we set

$$\rho(a) = \frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} \quad \text{and call} \quad \rho(H) = \sup\{\rho(a) \mid a \in H\} \quad \text{the } \textit{elasticity} \text{ of } H.$$

Note that H is half-factorial if and only if $\rho(H) = 1$.

For two factorizations z, $z' \in \mathsf{Z}(H)$, we call

$$\mathsf{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z,z')}\right|, \left|\frac{z'}{\gcd(z,z')}\right|\right\} \quad \text{the } \textit{distance} \text{ between } z \text{ and } z'.$$

Definition 2.1. Let H be an atomic monoid and $a \in H$.

1. Factorizations $z_0, \ldots, z_n \in \mathsf{Z}(a)$ with $n \in \mathbb{N}$ and $\mathsf{d}(z_{i-1}, z_i) \le N$ for some $N \in \mathbb{N}$ and $i \in [1, n]$ are called
   - an *N-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$).
   - a *monotone N-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| \le |z_i|$ for all $i \in [1, n]$.
2. The
   - *catenary degree* $\mathsf{c}(a)$
   - *monotone catenary degree* $\mathsf{c}_{\text{mon}}(a)$

   denotes the smallest $N \in \mathsf{N}_0 \cup \{\infty\}$ such that for all z, $z' \in \mathsf{Z}(a)$ there is
   - an N-chain concatenating z and z'.
   - a monotone N-chain concatenating z and z'.

   Then we call
   - $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\}$ the *catenary degree* of H.
   - $\mathsf{c}_{\text{mon}}(H) = \sup\{\mathsf{c}_{\text{mon}}(a) \mid a \in H\}$ the *monotone catenary degree* of H.

Note that $\mathsf{c}(H) \le \mathsf{c}_{\text{mon}}(H)$ and that equality holds if H is half-factorial by [21, Lemma 4.4.1].

Definition 2.2. Let H be a reduced atomic monoid.

1. For $a \in H$ and $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a, x)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
   If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.

   For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define

   $$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\, x \in X\},$$

   and we define $\mathsf{t}(H) = \mathsf{t}(H, \mathcal{A}(H))$. This is called the *tame degree* of H.
2. If $\mathsf{t}(H) < \infty$, then we call H *tame*

Here we recall the exact definitions from [21, Definition 2.3] for the $\mathcal{R}_{\text{eq}}$-relation and the $\mathcal{R}$-relation, the latter one coinciding with the one given in [20, Section 3].

Definition 2.3. Let H be a reduced atomic monoid and $a \in H$.

1. Factorizations $z_0, \ldots, z_n \in \mathsf{Z}(a)$ with $n \in \mathbb{N}$ and $\gcd(z_{i-1}, z_i) \ne 1$ for all $i \in [1, n]$ are called
   - an *$\mathcal{R}$-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$).
   - a *monotone $\mathcal{R}$-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| \le |z_i|$ for all $i \in [1, n]$.
   - an *equal-length $\mathcal{R}$-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| = |z_i|$ for all $i \in [1, n]$.
2. Two elements z, $z' \in \mathsf{Z}(H)$ are
   - *$\mathcal{R}$-related*
   - *$\mathcal{R}_{\text{eq}}$-related*

   if there is an
   - $\mathcal{R}$-chain
   - equal-length $\mathcal{R}$-chain

   concatenating z and z'. We then write $z \approx z'$ respectively $z \approx_{\text{eq}} z'$.

Note that with the above definitions $\approx$ and $\approx_{\text{eq}}$ are congruences on $\mathsf{Z}(H) \times \mathsf{Z}(H)$.

Definition 2.4. Let $H \subset D$ be monoids.

1. We call $H \subset D$ *saturated* or, equivalently, *a saturated submonoid* if, for all a, $b \in H$, $a \mid b$ in D already implies that $a \mid b$ in H.
2. If $H \subset D$ is a saturated submonoid, then we set $D/H = \{a\mathsf{q}(H) \mid a \in D\}$ and $[a]_{D/H} = a\mathsf{q}(H)$ and we call $\mathsf{q}(D)/\mathsf{q}(H) = \mathsf{q}(D/H)$ the *class group* of H in D.

Definition 2.5. Let H be an atomic monoid. We call

$$\sim_H = \{(x, y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi(x) = \pi(y)\} \qquad \text{the } \textit{monoid of relations} \text{ of } H,$$

$$\sim_{H,\text{mon}} = \{(x, y) \in \sim_H \mid |x| \le |y|\} \qquad \text{the } \textit{monoid of monotone relations} \text{ of } H,$$

and, for $a \in H$, we set

$$\mathcal{A}_a(\sim_H) = \mathcal{A}(\sim_H) \cap (\mathsf{Z}(a) \times \mathsf{Z}(a)),$$
$$\mathcal{A}_a(\sim_{H,\text{mon}}) = \mathcal{A}(\sim_{H,\text{mon}}) \cap (\mathsf{Z}(a) \times \mathsf{Z}(a)).$$

By [20, Lemma 11], $\sim_H \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ is a saturated submonoid of a free monoid and thus a Krull monoid by [12, Theorem 2.4.8.1]. Unfortunately, $\sim_{H,\mathrm{mon}} \subset \sim_H$ is not saturated.

Notions for integral domains. For an integral domain R, we set $R^\bullet = R \setminus \{0\}$ for the commutative, cancellative monoid of non-zero elements of R. Additionally, all notions, which were introduced for monoids, are used for domains, too; for example, we write $\mathcal{A}(R)$ instead of $\mathcal{A}(R^\bullet)$ for the set of atoms.

Definition 2.6. Let R be an integral domain and $K = \mathsf{q}(R)$ the quotient field of R.

1. We call $\mathrm{spec}(R)$ the set of all prime ideals of R.
2. We set
$$\mathfrak{X}(R) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid \mathfrak{p} \neq 0 \text{ and } \mathfrak{p} \text{ is minimal}\}$$
for the *set of minimal prime ideals* of R.
3. Let $L \supset K$ be a field extension. We call $b \in L$ *integral* over R if there is a monic polynomial $f \in R[X]$ such that $f(b) = 0$.
4. We call
$$\mathsf{cl}_L(R) = \{b \in L \mid b \text{ is integral over } R\} \quad \text{the } \textit{integral closure} \text{ of } R \text{ in } L$$
and we set $\bar{R} = \mathsf{cl}_K(R)$ for the integral closure of an integral domain (in its quotient field).
5. For non-empty subsets $X, Y \subset K$, we define
$$(Y : X) = (Y :_K X) = \{a \in K \mid aX \subset Y\} \text{ and } X^{-1} = (R : X).$$
We denote by $\mathcal{I}(R)$ the *set of all ideals* of R and we call an ideal $\mathfrak{a} \in \mathcal{I}(R)$ *invertible* if $\mathfrak{a}\mathfrak{a}^{-1} = R$. Then we denote by $\mathcal{I}^*(R)$ the *set of all invertible ideals* of R.

Definition 2.7. A one-dimensional noetherian domain R is called *locally half-factorial* if $\mathcal{I}^*(R)$ is half-factorial.
Note that this notion of being locally half-factorial does not coincide with the one defined in [2] but coincides with what is called purely locally half-factorial there.
By [12, Theorem 3.7.1], we have $\mathcal{I}^*(R) \cong \coprod_{\mathfrak{p} \in \mathfrak{X}(R)} (R_\mathfrak{p}^\bullet)_{\mathrm{red}}$. Thus $\mathcal{I}^*(R)$ is half-factorial if and only if $(R_\mathfrak{p}^\bullet)_{\mathrm{red}}$ is half-factorial for all $\mathfrak{p} \in \mathfrak{X}(R)$.

3. Proof of the main theorem

Before we can prove the main theorem, we need to gather some additional tools, among these the notion of T-block monoids over finite abelian groups, the concept of transfer homomorphism, and some monoid theoretic preliminaries. Once all these things at hand, we will exploit the results from [20] and [21] to give the final proof of the main theorem.

T-block monoids and transfer principles. First, we briefly fix the notation for T-block monoids, which are a generalization of the concept of block monoids, and therefore have their origin in zero-sum theory; for a detailed exposition of these aspects, the reader is referred to [12, Chapter 3] . Let G be an additively written finite abelian group, $G_0 \subset G$ a subset, and $\mathcal{F}(G_0)$ the free abelian monoid with basis G_0. The elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. If a sequence $S \in \mathcal{F}(G_0)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G_0$. For a sequence $S = g_1 \cdot \ldots \cdot g_l$, we call

$|S| = l$ the *length* of S and
$\sigma(S) = \sum_{i=1}^{l} g_i \in G$ the *sum* of S.

The sequence S is called a *zero-sum sequence* if $\sigma(S) = \mathbf{0}$. We set
$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \quad \text{for the } \textit{block monoid} \text{ over } G_0$$
and $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ for its set of atoms.

Then, the *Davenport constant* $\mathsf{D}(G_0) \in \mathbb{N}$ is defined to be the supremum of all lengths of sequences in $\mathcal{A}(G_0)$.

Now we are able to give the precise definition of T-block monoids.

Definition 3.1. Let G be an additive abelian group, T a monoid, $\iota : T \to G$ a homomorphism, and $\sigma : \mathcal{F}(G) \to G$ the unique homomorphism such that $\sigma(g) = g$ for all $g \in G$. Then we call
$$\mathcal{B}(G, T, \iota) = \{St \in \mathcal{F}(G) \times T \mid \sigma(S) + \iota(t) = \mathbf{0}\} \quad \text{the } T\textit{-block monoid} \text{ over } G \text{ defined by } \iota.$$
If $T = \{1\}$, then $\mathcal{B}(G, T, \iota) = \mathcal{B}(G)$ is the block monoid over G.

Next we give the transfer homomorphism and, then, we use it to transport questions on the arithmetic of our investigated monoids to T-block monoids.

Definition 3.2. A monoid homomorphism $\theta : H \to B$ is called a *transfer homomorphism* if it has the following properties:

T1 $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

T2 If $a \in H$, r, $s \in B$ and $\theta(a) = rs$, then there exist b, $c \in H$ such that $\theta(b) \sim r$, $\theta(c) \sim s$, and $a = bc$.

Definition 3.3. Let $\theta : H \to B$ be a transfer homomorphism of atomic monoids and $\bar{\theta} : \mathsf{Z}(H) \to \mathsf{Z}(B)$ the unique homomorphism satisfying $\bar{\theta}(uH^\times) = \theta(u)B^\times$ for all $u \in \mathcal{A}(H)$. We call $\bar{\theta}$ the extension of θ to the factorization monoids.
For $a \in H$, the *catenary degree in the fibers* $\mathsf{c}(a, \theta)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any two factorizations z, $z' \in \mathsf{Z}(a)$ with $\bar{\theta}(z) = \bar{\theta}(z')$, there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, $\bar{\theta}(z_i) = \bar{\theta}(z)$, and $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$; that is, z and z' can be concatenated by an N-chain in the fiber $\mathsf{Z}(a) \cap \bar{\theta}^{-1}((\bar{\theta}(z)))$.

Also, $\mathsf{c}(H, \theta) = \sup\{\mathsf{c}(a, \theta) \mid a \in H\}$ is called the *catenary degree in the fibers* of H.

Lemma 3.4. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, let $G = \mathsf{q}(D/H)$ be its class group, let $\iota : T \to G$ be a homomorphism defined by $\iota(t) = [t]_{D/H}$, and suppose each class in G contains some prime element from P. Then*

1. *The map $\beta : H \to \mathcal{B}(G, T, \iota)$, given by $\beta(pt) = [p]_{D/H} + \iota(t) = [p]_{D/H} + [t]_{D/H}$, is a transfer homomorphism onto the T-block monoid over G defined by ι, and $\mathsf{c}(H, \beta) \le 2$*
2. *The following inequalities hold:*
$$\begin{aligned} \mathsf{c}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{c}(H) && \le \max\{\mathsf{c}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\}, \\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{c}_{\mathrm{mon}}(H) && \le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\}, \text{ and} \\ \mathsf{t}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{t}(H) && \le \mathsf{t}(\mathcal{B}(G,T,\iota)) + \mathsf{D}(G) + 1. \end{aligned}$$
In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}(G,T,\iota))$ holds if $\mathsf{c}(\mathcal{B}(G,T,\iota)) \ge 2$, and the equality $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota))$ holds if $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \ge 2$.
3. *$\mathcal{L}(H) = \mathcal{L}(\mathcal{B}(G,T,\iota))$, $\triangle(H) = \triangle(\mathcal{B}(G,T,\iota))$, $\min \triangle(H) = \min \triangle(\mathcal{B}(G,T,\iota))$, and $\rho(H) = \rho(\mathcal{B}(G,t,\iota))$.*
4. *We set $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\}$. Then $\mathcal{B}$ and $\mathcal{B}(G,T,\iota)$ have the same arithmetical properties, and*
$$\begin{aligned} \mathsf{c}(\mathcal{B}) \le \quad & \mathsf{c}(H) && \le \max\{\mathsf{c}(\mathcal{B}), \mathsf{c}(H,\beta)\}, \\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \le \quad & \mathsf{c}_{\mathrm{mon}}(H) && \le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}), \mathsf{c}(H,\beta)\}, \text{ and} \\ \mathsf{t}(\mathcal{B}) \le \quad & \mathsf{t}(H) && \le \mathsf{t}(\mathcal{B}) + \mathsf{D}(G) + 1. \end{aligned}$$
In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B})$ holds if $\mathsf{c}(\mathcal{B}) \ge 2$, and the equality $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B})$ holds if $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \ge 2$.
Additionally, $\mathcal{L}(H) = \mathcal{L}(\mathcal{B})$, $\triangle(H) = \triangle(\mathcal{B})$, $\min \triangle(H) = \min \triangle(\mathcal{B})$, and $\rho(H) = \rho(\mathcal{B})$.

Proof.

1. Follows by [12, Proposition 3.2.3.3 and Proposition 3.4.8.2].
2. The assertion for the catenary degree follows by [12, Theorem 3.2.5.5], the assertion for the monotone catenary degree by [12, Lemma 3.2.6], and the assertion for the tame degree by [12, Theorem 3.2.5.1].
3. Follows by [12, Proposition 3.2.3.5].
4. Since $\mathbf{0} \in \mathcal{B}(G,T,\iota)$ is a prime element, it defines a partition $\mathcal{B}(G,T,\iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\}$. Thus all studied arithmetical invariants coincide for $\mathcal{B}$ and $\mathcal{B}(G,T,\iota)$. Now the assertions follow from part 2 and part 3. □

Lemma 3.5. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ its class group, and suppose each class in G contain some $p \in P$.*

1. *If $|G| \ge 3$, then $\min \triangle(H) = 1$, $\rho(H) > 1$, $\mathsf{c}(H) \ge 3$.*
2. *$\rho(H) \le \mathsf{D}(G)\rho(T)$.*

Proof. We define a homomorphism $\iota : T \to G$ by $\iota(t) = [t]_{D/H}$ and write $\mathcal{B}(G,T,\iota)$ for the T-block monoid over G defined by ι.

1. Then $\mathcal{B}(G) \subset \mathcal{B}(G,T,\iota)$ is a divisor-closed submonoid. By [12, Theorem 6.7.1.2], we have $\min \triangle(G) = 1$, and thus $\min \triangle(\mathcal{B}(G,T,\iota)) = 1$ and $\mathsf{c}(\mathcal{B}(G,T,\iota)) \geq 3$ by [12, Theorem 1.6.3]. Now the assertions follow by Lemma 3.4.2 and Lemma 3.4.3.
2. By [12, Proposition 3.4.7.5], we have $\rho(\mathcal{B}(G,T,\iota)) \leq \mathsf{D}(G)\rho(T)$. Now the assertion again follows by Lemma 3.4.2. $\square$

Definition 3.6. A monoid H is called *finitely primary* if there exist $s, k \in \mathbb{N}$ and a factorial monoid $F = [p_1, \ldots, p_s] \times F^\times$ with the following properties:

- $H \setminus H^\times \subset p_1 \cdot \ldots \cdot p_s F$,
- $(p_1 \cdot \ldots \cdot p_s)^k F \subset H$, and
- $(p_1 \cdot \ldots \cdot p_s)^i F \not\subset H$ for $i \in [0, k)$.

If this is the case, then we call H a *finitely primary monoid* of *rank* s and *exponent* k.
Note that this definition is slightly more restrictive than the one given in [12, Definition 2.9.1]. By [12, Theorem 2.9.2.1], we get $F = \widehat{H}$, and therefore $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times \subset \mathsf{q}(H)$.
Then, for $i \in [1, s]$, we denote by $\mathsf{v}_{p_i} : \mathsf{q}(H) \to \mathbb{Z}$ the p_i-adic valuation of $\mathsf{q}(H)$.
Now let $H \subset \widehat{H} = [p] \times \widehat{H}^\times$ be a finitely primary monoid of rank 1 and exponent k. Then we set $\mathcal{U}_i(H) = \{u \in \widehat{H}^\times \mid p^i u \in H\}$ for $i \in \mathbb{N}_0$.

As a first observation, we find

$$\mathcal{U}_i(H) = \begin{cases} H^\times & i = 0 \\ \widehat{H}^\times & i \geq k \end{cases} \quad \text{and} \quad \mathcal{U}_i(H)\mathcal{U}_j(H) \subset \mathcal{U}_{i+j}(H) \text{ for all } i, j \in \mathbb{N}_0.$$

Definition 3.7. Let $s \in \mathbb{N}$, $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$, $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$, and $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times$ be a finitely primary monoid of rank s and exponent $\max\{k_1, \ldots, k_s\}$.
Then H is a *monoid of type* $(\mathbf{e}, \mathbf{k})$ if

- $\mathsf{v}_{p_i}(H) = e_i\mathbb{N}_0 \cup \mathbb{N}_{\geq k_i}$ for all $i = [1, s]$ and
- $p_1^{k_1} \cdot \ldots \cdot p_s^{k_s} \widehat{H} \subset H$.

If $s = 1$, i.e., $\mathbf{e} = (e) \in \mathbb{N}$ and $\mathbf{k} = (k) \in \mathbb{N}$, then we say that H is a monoid of type (e, k) instead of $(\mathbf{e}, \mathbf{k})$.

Lemma 3.8. *Let $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times$ be a reduced finitely primary monoid of rank s and exponent k.*

1. *The following statements are equivalent:*
   (a) *H is half-factorial.*
   (b) *H is of rank 1 and $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$.*
   (c) *H is of rank 1 and $(\mathcal{U}_1(H))^l = \mathcal{U}_l(H)$ for all $l \in \mathbb{N}$.*
   If any of these conditions hold, then $\mathcal{A}(H) = \{p_1\varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$, $(\mathcal{U}_1(H))^k = \widehat{H}^\times$, and H is a monoid of type $(1, k)$.
2. *If H is a half-factorial monoid of type $(1, k)$ and $a_1, \ldots, a_{k+1}, b \in \mathcal{A}(H)$, then there are some $b_1, \ldots, b_k \in \mathcal{A}(H)$ such that $a_1 \cdot \ldots \cdot a_{k+1} = bb_1 \cdot \ldots \cdot b_k$.*
   In particular, $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) \leq \mathsf{t}(H) \leq k + 1$.

Proof.

1. **(a) ⇒ (b).** If H is of rank $s \geq 2$, then we find $\rho(H) = \infty$ by [12, Theorem 3.1.5.2 (b)]. Thus H is of rank 1. Now we prove $\#\mathsf{v}_{p_1}(\mathcal{A}(H)) = 1$. [Then the assertion follows since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{n\}$ with $n \geq 2$ implies $\mathsf{v}_{p_1}(H) = n\mathbb{N}_0 \not\supset \mathbb{N}_{\geq k}$, a contradiction.] Suppose $\#\mathsf{v}_{p_1}(\mathcal{A}(H)) > 1$. Let $n = \min \mathsf{v}_{p_1}(\mathcal{A}(H))$, $m \in \mathsf{v}_{p_1}(\mathcal{A}(H)) \setminus \{n\}$, and $\varepsilon, \eta \in \widehat{H}^\times$ be such that $p_1^n\varepsilon, p_1^m\eta \in \mathcal{A}(H)$. Now we find
$$(p_1^m\eta)^k = (p_1^n\varepsilon)^k(p_1^{(m-n)k}\varepsilon^{-k}\eta^k).$$
   On the left side there are k atoms and on the right side at least $k+1$—a contradiction to H being half-factorial.
   (b) ⇒ (a). Since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$, we have $\mathsf{L}(a) = \{\mathsf{v}_{p_1}(a)\}$, i.e., $\#\mathsf{L}(a) = 1$ for all $a \in H \setminus H^\times$. Therefore, H is half-factorial.
   (b) ⇒ (c). Since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$, we have $\mathcal{A}(H) = \{p_1 u \mid u \in \mathcal{U}_1(H)\}$. Thus, for all $l \in \mathbb{N}$, we have $\mathcal{U}_l(H) \subset (\mathcal{U}_1(H))^l$. Since we always have $(\mathcal{U}_1(H))^l \subset \mathcal{U}_l(H)$, the assertion follows.
   (c) ⇒ (b). Let $l \in \mathbb{N}_{\geq 2}$ and let $\varepsilon \in \mathcal{U}_l(H)$. By assumption, there are $\varepsilon_1, \ldots, \varepsilon_l \in \mathcal{U}_1(H)$ such that $(p_1\varepsilon_1) \cdot \ldots \cdot (p_1\varepsilon_l) = p_1^l\varepsilon$, and therefore $p_1^l\varepsilon \notin \mathcal{A}(H)$; thus $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$.
   Now we prove the additional statement. $\mathcal{A}(H) = \{p_1\varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$ has already been shown and $(\mathcal{U}_1(H))^k = \mathcal{U}_k(H) = \widehat{H}^\times$ is obvious. The last statement follows immediately by considering the definition of a monoid of type $(1, k)$; see Definition 3.7.

2. Let $H \subset [p_1] \times \widehat{H}^\times = \widehat{H}$ be a half-factorial monoid of type $(1,k)$ and let $a_1, \ldots, a_{k+1}, b \in \mathcal{A}(H)$. Since H is half-factorial, we have $\mathsf{c}(H) = \mathsf{c}_{\mathrm{mon}}(H)$ by [21, Lemma 4.4.1]. By part 1, we have $\mathcal{A}(H) = \{p_1\varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$. Then there are $\varepsilon_1, \ldots, \varepsilon_{k+1}, \eta \in \mathcal{U}_1(H)$ such that $a_i = p_1\varepsilon_i$ for $i \in [1, k+1]$ and $b = p_1\eta$. Now we find
$$a_1 \cdot \ldots \cdot a_{k+1} = (p_1\varepsilon_1) \cdot \ldots \cdot (p_1\varepsilon_{k+1}) = (p_1\eta)(p_1^k\eta^{-1}\varepsilon_1 \cdot \ldots \cdot \varepsilon_{k+1}).$$
By part 1, $(\mathcal{U}_1(H))^k = \widehat{H}^\times$, and thus there are $\eta_1, \ldots, \eta_k \in \mathcal{U}_1(H)$ such that $\eta^{-1}\varepsilon_1 \cdot \ldots \cdot \varepsilon_{k+1} = \eta_1 \cdot \ldots \cdot \eta_k$. Now we finish the proof by setting $b_i = p_1\eta_i$ for $i \in [1, k]$. $\square$

The result of Lemma 3.8.2 is sharp as the following example shows.

Example 3.9. Let $H \subset \widehat{H} = [p] \times \widehat{H}^\times$ be a half-factorial, reduced, finitely primary monoid of rank 1 and exponent $k-1$, with $k \geq 2$, such that $\widehat{H}^\times = \mathsf{C}_k^2 = \langle e_1 \rangle \times \langle e_2 \rangle$ and $\mathcal{U}_1(H) = \{1, e_1, e_2\}$. Then $\mathsf{c}(H) = k$.

Proof. By Lemma 3.8.2 we find $\mathsf{c}(H) \leq k$; thus the assertion follows from the equations
$$(pe_1)^k = (pe_2)^k = p^k \quad \text{and} \quad e_1^k = e_2^k = 1 \quad \text{and} \quad \mathrm{ord}(e_1) = \mathrm{ord}(e_2) = k,$$
since one cannot construct any shorter steps in between because of the minimality of the order of e_1 respectively e_2. $\square$

Definition 3.10 (cf. [12, Definition 3.6.3])**.** Let D be an atomic monoid.

1. If $H \subset D$ is an atomic submonoid, then we define
$$\rho(H, D) = \sup\left\{ \frac{\min \mathsf{L}_H(a)}{\min \mathsf{L}_D(a)} \,\middle|\, a \in H \setminus D^\times \right\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$
2. Let $H \subset D$ be a submonoid and $G_0 = \{[u]_{D/H} | u \in \mathcal{A}(D)\} \subset \mathsf{q}(D/H)$. We say that $H \subset D$ is *faithfully saturated* if H is atomic, $H \subset D$ is saturated and cofinal, $\rho(H, D) < \infty$, and $\mathsf{D}(G_0) < \infty$.

Lemma 3.11. *Let D be a half-factorial monoid and $H \subset D$ an atomic saturated submonoid. Then $\rho(H, D) \leq 1$.*

Proof. Let $\varepsilon \in D^\times \cap H$. Then $\varepsilon \mid 1$ in D, and thus $\varepsilon \mid 1$ in H, and therefore $\varepsilon \in H^\times$. Now we find $\rho(H, D) \leq \rho(D) = 1$ by [12, Proposition 3.6.6]. $\square$

Lemma 3.12. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D}_i^\times$ be reduced finitely primary monoids such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group, and let G be finite.*
Then

1. *D is a reduced BF-monoid.*
2. *$H \subset D$ is a faithfully saturated submonoid and H is also a reduced BF-monoid.*

Proof.

1. Since D is the direct product of reduced BF-monoids, D is a reduced BF-monoid.
2. Since, by part 1, D is a reduced BF-monoid, H is a reduced BF-monoid by [12, Proposition 3.4.5.5]. Since G and r are finite, $H \subset D$ is faithfully saturated by [12, Theorem 3.6.7]. $\square$

The following lemma offers a refinement of [12, Theorem 3.6.4] for faithfully saturated submonoids $H \subset D$ such that $\rho(H, D) = 1$. In our application, this new result yields a crucial refinement from $\mathsf{c}(H) \leq 6$ to $\mathsf{c}(H) \leq 4$.

Lemma 3.13. *Let D be a reduced atomic half-factorial monoid, $H \subset D$ a faithfully saturated submonoid with $\rho(H, D) = 1$, $G = \mathsf{q}(D/H)$ its class group, $\mathsf{D} = \mathsf{D}(G)$ its Davenport constant, and suppose each class in G contain some $u \in \mathcal{A}(D)$.*
Then
$$\mathsf{c}(H) \leq \max\left\{ \left\lfloor \frac{(\mathsf{D}+1)}{2} \mathsf{c}(D) \right\rfloor, \mathsf{D}^2 \right\}.$$

1. $\mathsf{c}(H) \leq \max\left\{ \left\lfloor \frac{(\mathsf{D}+1)}{2} \mathsf{c}(D) \right\rfloor, \mathsf{D}^2 \right\}$.
2. *If $a, c \in H$ and $x \in \mathsf{Z}_H(c)$, then*
$$\mathsf{t}_H(a, x) \leq |x| \left(1 + \mathsf{D}\frac{D-1}{2} \right) + \mathsf{D}\mathsf{t}_D(a, \mathsf{Z}_D(c)).$$

Proof. We start by developing the same machinery to compare the factorizations in H with those in D as in [12, Proof of Theorem 3.6.4]. Let $\pi_H : \mathsf{Z}(H) \to H$ and $\pi_D : \mathsf{Z}(D) \to D$ be the factorization homomorphisms and let $Y = \pi_D^{-1}(H) \subset \mathsf{Z}(D)$. Let $f : \mathsf{Z}(D) \to D/H$ be defined by $f(z) = [\pi_D(z)]_{D/H}$. Then f is an epimorphism and $Y = f^{-1}(0)$. Now [12, Proposition 2.5.1] implies that $Y \subset \mathsf{Z}(D)$ is saturated, that Y is a Krull monoid, and that f induces an isomorphism $f^* : \mathsf{Z}(D)/Y \to D/H$, since $Y \subset \mathsf{Z}(D)$ is cofinal. By [12, Theorem 3.4.10.5], we have $\mathsf{c}(Y) \le \mathsf{D}$, and by [12, Proposition 3.4.5.3] it follows that $|v| \le \mathsf{D}$ for all $v \in \mathcal{A}(Y)$. If $v \in Y$, then there exists a factorization $y \in \mathsf{Z}_H(\pi_D(v))$ such that $|y| \le |v|$.
If $\tilde z \in Y$ and $z \in \mathsf{Z}(H)$, then we say that z is induced by $\tilde z$ if $z = z_1 \cdot \ldots \cdot z_m$ and $\tilde z = \tilde z_1 \cdot \ldots \cdot \tilde z_m$, where $\tilde z_j \in \mathcal{A}(Y) \subset \mathsf{Z}(D)$, $z_j \in \mathsf{Z}_H(\pi_D(\tilde z_j))$ and $|z_j| \le |\tilde z_j|$ for all $j \in [1, m]$. If z is induced by $\tilde z$, then $\pi_H(z) = \pi_D(\tilde z)$ and $|z| \le |\tilde z|$. By definition, every factorization $\tilde z \in Y$ induces some factorization $z \in \mathsf{Z}(H)$. Also, if z is induced by $\tilde z$ and z' is induced by $\tilde z'$, then zz' is induced by $\tilde z \tilde z'$.
If $x = u_1 \cdot \ldots \cdot u_m \in \mathsf{Z}(H)$, where $u_j \in \mathcal{A}(H)$ and $\tilde u_j \in \mathsf{Z}_D(u_j)$, then $\tilde u_j \in \mathcal{A}(Y)$ and $|\tilde u_j| \le \mathsf{D}$ for all $j \in [1, m]$ by [12, Proposition 3.4.5.3]. Hence x is induced by $\tilde x = \tilde u_1 \cdot \ldots \cdot \tilde u_m$, and $|\tilde x| \le \mathsf{D}|x|$.
We prove the following assertions:

- **A0** Let $\tilde z \in Y$ with $\tilde z = a_1 \cdot \ldots \cdot a_m b_1 \cdot \ldots \cdot b_n$, where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}(H)$, $[a_1]_{D/H} = \ldots = [a_m]_{D/H} = \mathbf{0}_{D/H}$, and $[b_1]_{D/H}, \ldots, [b_n]_{D/H} \ne \mathbf{0}_{D/H}$. For any $z \in \mathsf{Z}(H)$ such that z is induced by $\tilde z$, we have $|z| = m + \left\lfloor \frac{n}{2} \right\rfloor$.
- **A1** For any $\tilde z$, $\tilde z' \in Y$, there exist z, $z' \in \mathsf{Z}(H)$ such that z is induced by $\tilde z$, z' is induced by $\tilde z'$, and $\mathsf{d}(z, z') \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\tilde z, \tilde z') \right\rfloor$.
- **A2** If $a \in H$, $\tilde z \in Y$, and z, $z' \in \mathsf{Z}_H(a)$ are both induced by $\tilde z$, then there exists a D^2-chain of factorizations in $\mathsf{Z}_H(a)$ concatenating z and z'.

Proof of A0. Let $\tilde z \in Y$ with $\tilde z = a_1 \cdot \ldots \cdot a_m b_1 \cdot \ldots \cdot b_n$, where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}(H)$, $[a_1]_{D/H} = \ldots = [a_m]_{D/H} = 0$, and $[b_1]_{D/H} = \ldots = [b_n]_{D/H} \ne 0_{D/H}$. Let now $z \in \mathsf{Z}(H)$ be induced by $\tilde z$. We have $a_i \in \mathcal{A}(H)$ for all $i \in [1, m]$ and—after renumbering if necessary—$b_1 \cdot \ldots \cdot b_{j_1}, b_{j_1+1} \cdot \ldots \cdot b_{j_2}, \ldots, b_{j_{k-1}+1} \cdot \ldots \cdot b_{j_k} \in \mathcal{A}(H)$ for some $k \in \mathbb{N}$ and $1 < j_1 + 1 < j_2 < j_2 + 1 < \ldots < j_{k-1} + 1 < j_k < n$ such that $a_1 \cdot \ldots \cdot a_m (b_1 \cdot \ldots \cdot b_{j_1})(b_{j_1+1} \cdot \ldots \cdot b_{j_2}) \cdot \ldots \cdot (b_{j_{k-1}+1} \cdot \ldots \cdot b_{j_k}) = z$. Then we have $|z| = m + \left\lfloor \frac{n}{2} \right\rfloor$. ☐

Proof of A1. Suppose that $\tilde z$, $\tilde z' \in Y$, $\tilde w = \gcd(\tilde z, \tilde z') \in \mathsf{Z}(D)$, $\tilde z = \tilde w \tilde y$, and $\tilde z' = \tilde w \tilde y'$, where $\tilde y$, $\tilde y' \in \mathsf{Z}(D)$. By [12, Proposition 3.4.5.6], there exists some $\tilde w_0 \in \mathsf{Z}(D)$ such that $\tilde w_0 \mid \tilde w$, $\tilde w_0 \tilde y \in Y$, and $|\tilde w_0| \le (\mathsf{D} - 1)|\tilde y|$. We may assume that there is no $a \in \mathcal{A}(D)$ with $a \mid \tilde w_0$ and $[a]_{D/H} = 0$. We set $\tilde w_1 = \tilde w_0^{-1} \tilde w$. Since $\tilde z = \tilde w_1(\tilde w_0 \tilde y) \in Y$ and $\tilde w_0 \tilde y \in Y$, we obtain $\tilde w_1 \in Y$, and since $\tilde z' = \tilde w_1(\tilde w_0 \tilde y') \in Y$ it follows that $\tilde w_0 \tilde y' \in Y$. Let v, u, $u' \in \mathsf{Z}(H)$ be such that v is induced by $\tilde w_0^{-1} \tilde w$, u is induced by $\tilde w_0 \tilde y$ and u' is induced by $\tilde w_0 \tilde y'$. Then $z = uv$ is induced by $\tilde w \tilde y = \tilde z$, $z' = u'v$ is induced by $\tilde w \tilde y' = \tilde z'$, and, by part **A1**,

$$\mathsf{d}(z, z') \le \max\{|u|, |u'|\} \le \max\{|\tilde y|, |\tilde y'|\} + \left\lfloor \frac{|\tilde w_0|}{2} \right\rfloor \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\tilde z, \tilde z') \right\rfloor . \qquad ☐$$

Proof of A2. For every $\tilde v \in \mathcal{A}(Y)$, we fix a factorization $\tilde v^* \in \mathsf{Z}(H)$ which is induced by $\tilde v$, and, for $\bar y = \tilde v_1 \cdot \ldots \cdot \tilde v_s \in \mathsf{Z}(Y)$, we set $\bar y^* = \tilde v_1^* \cdot \ldots \cdot \tilde v_s^* \in \mathsf{Z}(H)$. Then $\bar y^*$ is induced by $\pi_Y(\bar y)$, $|\bar y^*| \le |\pi_Y(\bar y)| \le \mathsf{D}|\bar y|$, and if $\bar y_1$, $\bar y_2 \in \mathsf{Z}(Y)$, then $\mathsf{d}(\bar y_1^*, \bar y_2^*) \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\bar y_1, \bar y_2) \right\rfloor$ by **A1**.
Let now z, $z' \in \mathsf{Z}_H(a)$ be both induced by $\tilde z$. Then $\tilde z = \tilde v_1 \cdot \ldots \cdot \tilde v_r = \tilde v_1' \cdot \ldots \cdot \tilde v_{r'}'$, $z = v_1 \cdot \ldots \cdot v_r$, and $z' = v_1' \cdot \ldots \cdot v_{r'}'$, where $\tilde v_i$, $\tilde v_i' \in \mathcal{A}(Y)$, v_i is induced by $\tilde v_i$, and v_i' is induced by $\tilde v_i'$. Since $\bar y = \tilde v_1 \cdot \ldots \cdot \tilde v_r \in \mathsf{Z}_Y(\tilde z)$, $\bar y' = \tilde v_1' \cdot \ldots \cdot \tilde v_{r'}' \in \mathsf{Z}_Y(\tilde z)$, and $\mathsf{c}(Y) \le \mathsf{D}$, there exists a D-chain $\bar y = \bar y_0, \bar y_1, \ldots, \bar y_l = \bar y'$ in $\mathsf{Z}_Y(\tilde z)$ concatenating $\bar y$ and $\bar y'$ in $\mathsf{Z}_Y(\tilde z)$. Then $\bar y_0^*, \bar y_1^*, \ldots, \bar y_l^*$ is a D-chain in $\mathsf{Z}_H(a)$ concatenating $\bar y^*$ and $\bar y'^*$. We have $\bar y_0^* = \tilde v_1^* \cdot \ldots \cdot \tilde v_r^*$, $z = v_1 \cdot \ldots \cdot v_r$, and since both v_i and v_i^* are induced by $\tilde v_i$, it follows that $\max\{|v_i|, |v_i^*|\} \le |\tilde v_i| \le \mathsf{D}$. For $i \in [0, r]$, we set $z_i = \tilde v_1^* \cdot \ldots \cdot \tilde v_i^* v_{i+1} \cdot \ldots \cdot v_r \in \mathsf{Z}_H(a)$. Then $z = z_0, z_1, \ldots, z_r = \bar y^*$ is a D^2-chain concatenating z and $\bar y^*$. In the same way, we get a D-chain concatenating $\bar y'^*$ and z'. Connecting these three chains, we get a D^2-chain in $\mathsf{Z}_H(a)$ concatenating z and z'. ☐

1. Assume $a \in H$ and z, $z' \in \mathsf{Z}_H(a)$. Let $\tilde z$, $\tilde z' \in Y$ be such that z is induced by $\tilde z$ and z' is induced by $\tilde z'$. Then $\tilde z$, $\tilde z' \in \mathsf{Z}_D(a)$, and therefore there exists a $\mathsf{c}(D)$-chain $\tilde z = \tilde z_0, \tilde z_1, \ldots, \tilde z_l = \tilde z_l'$ in $\mathsf{Z}_D(a)$. For $i \in [0, l-1]$, **A1** gives the existence of factorizations z_i', $z_i'' \in \mathsf{Z}_H(a)$ such that z_i' is induced by $\tilde z_i$, z_i'' is induced by $\tilde z_{i+1}$, and $\mathsf{d}(z_i', z_i'') \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{c}(D) \right\rfloor$. By **A2**, there exist D^2-chains of factorizations in $\mathsf{Z}_H(a)$ concatenating z and z_0', z_i'' and z_{i+1}' for all $i \in [0, l-1]$, and z_{l-1} and z'. Connecting all these chains, we obtain a $\max\left\{ \left\lfloor \frac{(\mathsf{D}+1)}{2} \mathsf{c}(D) \right\rfloor, \mathsf{D}^2 \right\}$-chain concatenating z and z'.

2. Suppose that a, $c \in H$, $x \in \mathsf{Z}_H(c)$, $z \in \mathsf{Z}_H(a)$, and $\mathsf{Z}_H(a) \cap x\mathsf{Z}(H) \neq \emptyset$. We set $\mathsf{t} = \mathsf{t}_D(a, \mathsf{Z}_D(c))$, and we must prove that there exists some $z' \in \mathsf{Z}_H(a) \cap x\mathsf{Z}(H)$ such that
$$\mathsf{d}(z, z') \le |x| \left(1 + \mathsf{D}\frac{D-1}{2}\right) + \mathsf{D}\mathsf{t}.$$
Let $\tilde{x} \in Y$ be such that x is induced by $\tilde{x}$ and $|\tilde{x}| \le \mathsf{D}|x|$. Suppose that $z = u_1 \cdot \ldots \cdot u_m$ and $\tilde{z} = \tilde{u}_1 \cdot \ldots \cdot \tilde{u}_m$, where $u_j \in \mathcal{A}(H)$ and $\tilde{u}_j \in \mathsf{Z}_D(u_j)$ for all $j \in [1, m]$. Then z is induced by $\tilde{z}$. Since $\mathsf{Z}_H(a) \cap x\mathsf{Z}(H) \neq \emptyset$, we obtain $\pi_D(\tilde{x}) = \pi_H(x) \mid a$, hence $\mathsf{Z}_D(a) \cap \tilde{x}\mathsf{Z}(D) \neq \emptyset$, and therefore there exists some $\tilde{z}' \in \mathsf{Z}_D(a) \cap \tilde{x}\mathsf{Z}(D)$ such that $\mathsf{d}(\tilde{z}, \tilde{z}') \le \mathsf{t}_D(a, \tilde{x}) \le \mathsf{t}$. After renumbering (if necessary), we may assume that
$$\gcd(\tilde{z}, \tilde{z}') = \prod_{j=1}^{k} \tilde{u}_j \prod_{j=k+1}^{m} y_j, \quad \text{and we set} \quad \tilde{y}' = \tilde{z}' \prod_{j=1}^{k} \tilde{u}_j^{-1},$$
where $k \in [0, m]$, $y_j \in \mathsf{Z}(D)$, $y_j \mid \tilde{u}_j$, $y_j \neq \tilde{u}_j$, and thus $|y_j| \le |\tilde{u}_j| - 1 \le \mathsf{D} - 1$ for all $j \in [k+1, m]$. Hence we obtain
$$\mathsf{t} \ge \mathsf{d}(\tilde{z}, \tilde{z}') = \mathsf{d}\left(\prod_{j=k+1}^{m} \tilde{u}_j y_j^{-1}, \tilde{y}' \prod_{j=k+1}^{m} y_j^{-1}\right) \ge \max\{m-k, |\tilde{y}'| - (m-k)(\mathsf{D}-1)\},$$
and therefore $|\tilde{y}'| \le \mathsf{t} + (m-k)(\mathsf{D}-1) \le \mathsf{t}\mathsf{D}$. After renumbering again (if necessary), we may suppose that $\tilde{x}_1 = \gcd(\tilde{u}_1 \cdot \ldots \cdot \tilde{u}_k, \tilde{x}) = y'_{l+1} \cdot \ldots \cdot y'_k$, where $l \in [0, k]$, $y'_j \in \mathsf{Z}(D)$ and $1 \neq y'_j \mid \tilde{u}_j$ for all $j \in [l+1, k]$. Then we have $k - l \le |\tilde{x}_1| \le |\tilde{x}| \le \mathsf{D}|x|$.

Since $\tilde{x} \mid \tilde{z}'$, it follows that $\tilde{x}_1^{-1}\tilde{x} \mid \tilde{x}_1^{-1}\tilde{z}' = \tilde{y}'(\tilde{x}_1^{-1}\tilde{u}_1 \cdot \ldots \cdot \tilde{u}_k)$, and since $\gcd(\tilde{x}_1^{-1}\tilde{u}_1 \cdot \ldots \cdot \tilde{u}_k, \tilde{x}_1^{-1}\tilde{x}) = 1$, we deduce $\tilde{x}_1^{-1}\tilde{x} \mid \tilde{y}'$. Hence $\tilde{x} \mid \tilde{y}'\tilde{x}_1 \mid \tilde{y}'\tilde{u}_{l+1} \cdot \ldots \cdot \tilde{u}_k$, and we set
$$\tilde{y} = \tilde{x}^{-1}\tilde{y}'\tilde{u}_{l+1} \cdot \ldots \cdot \tilde{u}_k = (\tilde{x}\tilde{u}_1 \cdot \ldots \cdot \tilde{u}_l)^{-1}\tilde{z}' \in \mathsf{Z}(D).$$
Since $\tilde{z}'$, $\tilde{x}$, $\tilde{u}_1, \ldots, \tilde{u}_l \in Y$ and $Y \subset \mathsf{Z}(D)$ is a saturated submonoid, we get $\tilde{y} \in Y$. Now we set $\tilde{y} = \tilde{y}_1\tilde{y}_2$ with $\tilde{y}_1 = (\tilde{x}_1^{-1}\tilde{x})^{-1}\tilde{y}'$ and $\tilde{y}_2 = (\tilde{u}_{l+1}y'^{-1}_{l+1}) \cdot \ldots \cdot (\tilde{u}_k y'^{-1}_k)$. Let $y \in \mathsf{Z}(H)$ be induced by $\tilde{y}$. then $z' = xyu_1 \cdot \ldots \cdot u_l \in \mathsf{Z}_H(a) \cap x\mathsf{Z}(H)$ is induced by $\tilde{z}'$, and $\mathsf{d}(z, z') = \mathsf{d}(u_{l+1} \cdot \ldots \cdot u_m, xy) \le \max\{m - l, |x| + |y|\}$. Now we start by computing $|y|$. Since, for all $j \in [l+1, k]$, there is no $a \in \mathcal{A}(D)$ such that $[a]_{D/H} = \mathbf{0}_{D/H}$ and $a \mid \tilde{u}_j y_j^{-1}$, we find using part **A0**
$$|y| \le |\tilde{y}_1| + \left\lfloor \frac{|\tilde{y}_2|}{2} \right\rfloor \le |\tilde{y}'| + \left\lfloor \frac{(k-l)(\mathsf{D}-1)}{2} \right\rfloor \le \mathsf{t}\mathsf{D} + \frac{\mathsf{D}(\mathsf{D}-1)}{2}|x| = \mathsf{D}\left(\mathsf{t} + \frac{\mathsf{D}-1}{2}|x|\right).$$
Furthermore, we have
$$m - l = (m-k) + (k-l) \le \mathsf{t} + \mathsf{D}|x|, \quad |x| + |y| \le |x| + \mathsf{D}\left(\mathsf{t} + \frac{\mathsf{D}-1}{2}|x|\right),$$
and $\mathsf{D} \ge 2$ implies
$$\mathsf{t} + \mathsf{D}|x| \le |x| \left(1 + \mathsf{D}\frac{D-1}{2}\right) + \mathsf{D}\mathsf{t}.$$
Hence we obtain the asserted bound for $\mathsf{d}(z, z')$. □

Lemma 3.14. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D}_i^\times$ be reduced half-factorial but not factorial monoids of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group, and suppose G is finite with each class in G containing some $p \in P$.*
Then

1. *$2 \le \mathsf{c}(D) = \max\{\mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\} \le \max\{k_1, \ldots, k_r\} + 1 \le 3$ and D is half-factorial.*
   In particular, $\mathsf{c}(D) = 2$ and $\mathsf{t}(D) = 2$ if $k_1 = \ldots = k_r = 1$.
2. *If $|G| = 1$, then $\mathsf{c}(H) = \mathsf{c}(D)$, $\mathsf{t}(H) = \mathsf{t}(D)$, and H is half-factorial.*
3. *If $|G| \ge 3$, then $(\mathsf{D}(G))^2 \ge \mathsf{c}(H) \ge 3$ and $\min \triangle(H) = 1$.*
4. *If $|G| = 2$, then $\mathsf{c}(H) \le 4$ and $\rho(H) \le 2$.*

Proof.

1. By Lemma 3.12.1, D is atomic. Trivially, we have $\mathsf{c}(\mathcal{F}(P)) = 0$. By Lemma 3.8.2 and the fact that D_i is not factorial, we find $2 \le \mathsf{c}(D_i) \le k_i + 1 \le 3$ for all $i \in [1, r]$. By [12, Proposition 1.6.8], we

find

$$\begin{aligned}\mathsf{c}(D) &= \max\{\mathsf{c}(\mathcal{F}(P)), \mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\} \\ &= \max\{\mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\} \\ &= \max\{k_1, \ldots, k_r\} + 1 \le 3.\end{aligned}$$

Thus the first part of the assertion follows. Since D is the direct product of half-factorial monoids, D is half-factorial by [12, Proposition 1.4.5]. We have $\mathsf{t}(D_i) = 2$ if $k_i = 1$ for all $i \in [1,r]$ by Lemma 3.8.2. Now $\mathsf{t}(D) = 2$ follows by [12, Proposition 1.6.8].

2. Here we have $H = D$ and thus the assertion follows from part 1.

Before the proof of the two remaining parts, we make the following observations. By Lemma 3.12.2, H is atomic, $H \subset D$ is a faithfully saturated submonoid, and, by Lemma 3.11, we have $\rho(H,D) \le 1$.

3. By part 1, we have $\mathsf{c}(D) \le 3$, by Lemma 3.5.1, we have $\min \triangle(H) = 1$, and, by [12, Lemma 1.4.9.2], we have $\mathsf{D}(G) \ge 3$. Using [12, Theorem 3.6.4.1], we find
$$3 \le \mathsf{D}(G) \le \mathsf{c}(H) \le \rho(H,D) \max\{\mathsf{c}(D), \mathsf{D}(G)\}\mathsf{D}(G) = (\mathsf{D}(G))^2.$$
4. Since $|G| = 2$, we have $\mathsf{D}(G) = 2$, and since $D_1 \times \ldots \times D_r$ is half-factorial, i.e., $\rho(D_1 \times \ldots \times D_r) = 1$, we find $\rho(H) \le 2$ by Lemma 3.5.2. When we apply Lemma 3.13.1, we find
$$\mathsf{c}(H) \le \max\left\{\left\lfloor (\mathsf{D}(G)+1)\frac{\mathsf{c}(D)}{2}\right\rfloor, \mathsf{D}(G)^2\right\} \le \left\{\left\lfloor\frac{9}{2}\right\rfloor, 4\right\} = 4. \qquad \square$$

For the rest of this section, we define additional shorthand notation. Let D be a monoid, $P \subset D$ a set of prime elements, and $T \subset D$ a submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H) = \mathsf{q}(D)/\mathsf{q}(H)$ its class group, suppose each $g \in G$ contains some $p \in P$, and let $\mathcal{B}(G,T,\iota)$ be the T-block monoid over G defined by the homomorphism $\iota : T \to G$, $\iota(t) = [t]_{D/H}$.
For a subset $S \subset \mathcal{B}(G,T,\iota)$ and an element $g \in G$, we set $S_g = S \cap \iota^{-1}(\{g\})$.

Lemma 3.15. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D_i}^{\times}$ be reduced half-factorial monoids of type $(1,k_i)$ with $k_i \in \{1,2\}$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group with $|G| = 2$, say $G = \{\mathbf{0}, g\}$, suppose each class in G contains some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$.*
Then we find the following for the atoms of the $(D_1 \times \ldots \times D_r)$-block monoid over G defined by ι, i.e., $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$:

$$\begin{aligned}&\mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)) \\ &= \{\mathbf{0}, g^2\} \\ &\cup \{p_i\varepsilon \mid i \in [1,r],\, \varepsilon \in \mathcal{U}_1(D_i),\, \iota(p_i\varepsilon) = \mathbf{0}\} \\ &\cup \{p_i\varepsilon g \mid i \in [1,r],\, \varepsilon \in \mathcal{U}_1(D_i),\, \iota(p_i\varepsilon) = g\} \\ &\cup \{p_i^2\varepsilon \mid i \in [1,r],\, \varepsilon \in (\widehat{D_i}^{\times})_{\mathbf{0}} \setminus (\mathcal{U}_1(D_i)_{\iota(p_i)})^2\} \\ &\cup \{p_ip_j\varepsilon_i\varepsilon_j \mid i,\, j \in [1,r],\, i \ne j,\, \varepsilon_i \in \mathcal{U}_1(D_i),\, \varepsilon_j \in \mathcal{U}_1(D_j),\, \iota(p_i\varepsilon_i) = \iota(p_j\varepsilon_j) = g\}\,.\end{aligned}$$

Proof. For short, we write $\mathcal{B} = \mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$. Since $|G| = 2$, we have $\mathsf{D}(G) = 2$, and thus every atom of $\mathcal{B}$ is a product of at most two atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$. First, we write down all atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$, namely,

$$\mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) = \{\mathbf{0}, g\} \cup \bigcup_{i \in [1,r]} \{p_i\varepsilon \mid \varepsilon \in \mathcal{U}_1(D_i)\}\,,$$

by Lemma 3.8.1. Now, we find

$$\begin{aligned}\mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) \cap \mathcal{B} &= \{\mathbf{0}\} \cup \{p_i\varepsilon \mid i \in [1,r],\, \varepsilon \in \mathcal{U}_1(D_i),\, \iota(p_i\varepsilon) = \mathbf{0}\}\,, \text{ and} \\ \mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) \setminus \mathcal{B} &= \{\mathbf{g}\} \cup \{p_i\varepsilon \mid i \in [1,r],\, \varepsilon \in \mathcal{U}_1(D_i),\, \iota(p_i\varepsilon) = g\}\,.\end{aligned}$$

By Lemma 3.12, D and H are reduced, and therefore $\varepsilon_i\varepsilon_j \notin \mathcal{B}$ for all $i,\, j \in [1,r]$, $i \ne j$, $\varepsilon_i \in \mathcal{U}_1(D_i)$, and $\varepsilon_j \in \mathcal{U}_1(D_i)$. Thus the following products of two atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$ are atoms of $\mathcal{B}$:

$$\begin{aligned}\mathcal{A}(\mathcal{B}) \supset &\{g^2\} \\ &\cup \{p_i\varepsilon g \mid i \in [1,r],\, \varepsilon \in \mathcal{U}_1(D_i),\, \iota(p_i\varepsilon) = g\} \\ &\cup \{p_i^2\varepsilon \mid i \in [1,r],\, \varepsilon \in (\widehat{D_i}^{\times})_{\mathbf{0}} \setminus (\mathcal{U}_1(D_i)_{\iota(p_i)})^2\} \\ &\cup \{p_ip_j\varepsilon_i\varepsilon_j \mid i,\, j \in [1,r],\, i \ne j,\, \varepsilon_i \in \mathcal{U}_1(D_i),\, \varepsilon_j \in \mathcal{U}_1(D_j),\, \iota(p_i\varepsilon_i) = \iota(p_j\varepsilon_j) = g\}\,.\end{aligned}$$

Since we have run through all possible combinations, the assertion follows. □

Lemma 3.16. *Let $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$ be a monoid, where $P \subset D$ is a set of prime elements, $r \in \mathbb{N}$, and, for all $i \in [1, r]$, $D_i \subset [p_i] \times \widehat{D_i}^\times$ is a reduced half-factorial but not factorial monoid of type $(1, 1)$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group with $|G| = 2$, and suppose each class in G contains some $p \in P$. Let $\iota : D_1 \times \ldots \times D_r \to G$ be defined by $\iota(t) = [t]_{D/H}$, denote by $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ the $(D_1 \times \ldots \times D_r)$-block monoid over G defined by ι, and set $|\cdot|_\mathcal{B} = |\cdot|_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}$.*

1. *If $(x, y) \in \sim_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}$ with $|y|_\mathcal{B} \geq |x|_\mathcal{B}$ and $|y|_\mathcal{B} \geq 5$, then there is a monotone $\mathcal{R}$-chain concatenating x and y; in particular, $x \approx y$, and if $|x|_\mathcal{B} = |y|_\mathcal{B}$, then $x \approx_{\mathrm{eq}} y$.*
2. *Additionally,*
$$\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) \leq \sup\{|y|_\mathcal{B} \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}), |x|_\mathcal{B} \leq |y|_\mathcal{B} \leq 4\}.$$

Proof. Let $|G| = 2$, say $G = \{\mathbf{0}, g\}$. By Lemma 3.4.4, we set $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) \mid \mathbf{0} \nmid S\}$. Before we start the actual proof, we establish some machinery to deal with factorizations in $\mathcal{B}$ and their lengths more systematically.

We set $D_0 = \mathcal{F}(\{g\})$, whence $\mathcal{A}(D_0) = \{g\}$ and $\mathsf{Z}(D_0) = D_0$. We define $\iota : D_0 \to G$ by $\iota(g^k) = kg$ for all $k \in \mathbb{Z}$. For $i \in [0, r]$, let $\pi_i : \mathsf{Z}(D_i) \to D_i$ be the factorization homomorphism. We set $D' = D_0 \times \ldots \times D_r$ and obtain $\mathcal{A}(D') = \mathcal{A}(D_0) \cup \ldots \cup \mathcal{A}(D_r)$. If $a = a_0 \cdot \ldots \cdot a_r \in D'$, where $a_i \in D_i$ for all $i \in [0, r]$, then we set $\iota(a) = \iota(a_0) + \iota(a_1 \cdot \ldots \cdot a_r) = \iota(a_0) + \ldots + \iota(a_r)$. Then $\iota : D' \to G$ is a homomorphism and $\mathcal{B} = \iota^{-1}(\mathbf{0}) \subset D'$ is a saturated submonoid, whose atoms are given by the following assertion **A1**.

A1 An element $x \in D_0 \times \cdot \ldots \cdot \times D_r$ is an atom of $\mathcal{B}$ if and only if it is of one of the following forms:
- $x = a \in \mathcal{A}(D_i)$ for some $i \in [1, r]$ and $\iota(a) = \mathbf{0}$.
- $x = a_1a_2$, where $a_1 \in \mathcal{A}(D_i)$, $a_2 \in \mathcal{A}(D_j)$, for some $i, j \in [0, r]$, $i \neq j$, and $\iota(a_1) = \iota(a_2) = g$.
- $x = a_1a_2$, where $a_1, a_2 \in \mathcal{A}(D_i)$ for some $i \in [0, r]$ such that $\iota(v) = g$ for all $v \in \mathcal{A}(D_i)$.

We will call the atoms of the third form *pure in i*.

Proof of **A1**. By the listing of all atoms of $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ in Lemma 3.15 and the fact that $\mathcal{A}(\mathcal{B}) = \mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)) \setminus \{\mathbf{0}\}$, we must only show the last statement in the case $i \in [1, r]$. Suppose there are $a_1, a_2 \in \mathcal{A}(D_i)$ such that $a = a_1a_2 \in \mathcal{A}(\mathcal{B})$. Then, obviously, $\iota(a_1) = \iota(a_2) = g$. Now we assume there is some $v \in \mathcal{A}(D_i)$ with $\iota(v) = \mathbf{0}$. By Lemma 3.8.2, there is $v' \in \mathcal{A}(D_i)$ such that $a_1a_2 = vv'$, and then $\iota(v') = \mathbf{0}$, a contradiction. □

Let $F = \mathsf{Z}(D') = \mathsf{Z}(D_0) \times \ldots \times \mathsf{Z}(D_r) = \mathcal{F}(\mathcal{A}(D'))$ be the factorization monoid of D'. Then $\pi = \pi_0 \times \ldots \times \pi_r : F \to D'$ is the factorization homomorphism of D'. We denote by $|\cdot| = |\cdot|_F$ the length function in the free monoid F, and for $x, y \in F$, we write $x \mid y$ instead of $x \mid_F y$. For $a \in \mathcal{A}(\mathcal{B})$, let $\theta_0(a) \in \pi^{-1}(a) \subset \mathsf{Z}(D')$ be a factorization of a in D'. If $a \in \mathcal{A}(D')$, then $\theta_0(a) = a$; otherwise $\theta_0(a) = a_1a_2 \in F$ for some $a_1, a_2 \in \mathcal{A}(D')$ such that $a = a_1a_2$ in D'. By **A1**, $\#\pi^{-1}(a) = 1$ unless a is pure in i for some $i \in [1, r]$. Let $\theta : \mathsf{Z}(\mathcal{B}) \to F$ be the unique monoid homomorphism satisfying $\theta|\mathcal{A}(\mathcal{B}) = \theta_0$. Then θ induces the following commutative diagram

$$\begin{tikzcd}
\mathsf{Z}(\mathcal{B}) \arrow[r, "\theta"] \arrow[d, "\pi_\mathcal{B}"] & F = \mathsf{Z}(D') \arrow[d, "\pi_{D'}"] \\
\mathcal{B} \arrow[r, hook] & D',
\end{tikzcd}$$

where $\pi_\mathcal{B}$ denotes the factorization homomorphism of $\mathcal{B}$ and the bottom arrow denotes the inclusion. For $x \in \mathsf{Z}(\mathcal{B})$, we set $|x| = |\theta(x)|$. For $x \in \mathsf{Z}(\mathcal{B})$, we define its components $x_i \in \mathsf{Z}(D_i)$ for $i \in [0, r]$ by $\theta(x) = x_0 \cdot \ldots \cdot x_r$. Then $\pi \circ \theta(x) \in \mathcal{B}$ implies $\iota \circ \pi_0(x_0) + \ldots + \iota \circ \pi_r(x_r) = \mathbf{0}$. For $i \in [0, r]$, we set $x_i = u_{i,1} \cdot \ldots \cdot u_{i,k_i} v_{i,1} \cdot \ldots \cdot v_{i,l_i}$, where $u_{i,j}, v_{i,j} \in \mathcal{A}(D_i)$, $\iota(u_{i,j}) = \mathbf{0}$, and $\iota(v_{i,j}) = g$. We define $x_i', x_i'' \in \mathsf{Z}(D_i)$ by $x_i' = u_{i,1} \cdot \ldots \cdot u_{i,k_i}$ and $x_i'' = v_{i,1} \cdot \ldots \cdot v_{i,l_i}$, whence $x_i = x_i'x_i''$. In particular, $|x_0'| = 0$, $x_0 = x_0''$, and $\iota \circ \pi_i(x_i) = l_ig = |x_i''|g$. Therefore we obtain $|x_0''| + \ldots + |x_r''| \equiv 0 \mod 2$. If $i \in [1, r]$ and $a \in D_i$ is such that $a \mid x_i'$, then $a \mid_\mathcal{B} x$. In $\mathsf{Z}(\mathcal{B})$, there is a factorization $x = u_1 \cdot \ldots \cdot u_mv_1 \cdot \ldots \cdot v_n$, where $u_j, v_j \in \mathcal{A}(\mathcal{B})$, $|u_j| = 1$ for all $j \in [1, m]$, $|v_j| = 2$ for all $j \in [1, n]$, and we obtain

$$m = \sum_{i=1}^{r} |x_i'|, \quad n = \frac{1}{2}\sum_{i=0}^{r} |x_i''|, \quad \text{and} \quad |x|_\mathcal{B} = m + n = \frac{1}{2}\sum_{i=0}^{r}(|x_i| + |x_i'|) \leq \sum_{i=0}^{r} |x_i|.$$

Assume now that $x = x_0 \cdot \ldots \cdot x_r$, $y = y_0 \cdot \ldots \cdot y_r \in \mathsf{Z}(\mathcal{B})$ are as above, and suppose that $(x, y) \in \sim_\mathcal{B}$. Then $x_0 = y_0$, $|x_i| = |y_i|$ (since each D_i is half-factorial), $\pi_i(x_i) = \pi_i(y_i) \in D_i$, and thus $\iota \circ \pi_i(x_i) = \iota \circ \pi_i(y_i) \in G$, and therefore $|x_i''| \equiv |y_i''| \mod 2$ and $|x_i'| \equiv |y_i'| \mod 2$ for all $i \in [1, r]$. Consequently, it follows that the following are all equivalent:

- $|x|_\mathcal{B} \leq |y|_\mathcal{B}$

- $\sum_{i=1}^{r} |x_i'| \le \sum_{i=1}^{r} |y_i'|$
- $\sum_{i=1}^{r} |x_i''| \ge \sum_{i=1}^{r} |y_i''|$

Additionally, we find

$$2|x|_{\mathcal{B}} = \sum_{i=0}^{r} (|x_i| + |x_i'|) \ge \sum_{i=0}^{r} (|y_i|) \ge |y|_{\mathcal{B}},$$

and thus $|y|_{\mathcal{B}} \ge 5$ implies $|x|_{\mathcal{B}} \ge 3$.
Before we start with the actual proof of part 1 of Lemma 3.16, we prove the following reduction step.

A2 In the proof of part 1 of Lemma 3.16, we may assume that $|x_i| = |y_i| \ge 2$ for all $i \in [1, r]$.

Proof of **A2**. If $i \in [1, r]$, then $|x_i| = 0$ if and only if $|y_i| = 0$, and in this case we may neglect this component. If $|x_i| = 0$ for all $i \in [1, r]$, then there is nothing to do. If $i \in [1, r]$, then $|x_i| = 1$ if and only if $|y_i| = 1$, and then $x_i = y_i \in \mathcal{A}(D_i)$. Suppose that $i \in [1, r]$ and $|x_i| = 1$. If $\iota(x_i) = \mathbf{0}$, then $x_i \in \mathcal{A}(\mathcal{B})$ and x_i is a greatest common divisor of x and y in $\mathsf{Z}(\mathcal{B})$; hence (x, y) is a monotone $\mathcal{R}$-chain concatenating x and y. If $\iota(x_i) = g$, then we set $\tilde{x} = g x_i^{-1} x$, $\tilde{y} = g y_i^{-1} y$, and then $(\tilde{x}, \tilde{y}) \in \sim_{\mathcal{B}}$, $|\tilde{x}_i| = |\tilde{y}_i| = 0$, and whenever there is a monotone $\mathcal{R}$-chain concatenating $\tilde{x}$ and $\tilde{y}$, then there is a monotone $\mathcal{R}$-chain concatenating x and y. $\square$

Now we are ready to do the actual proof of the lemma. Suppose that $(x, y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \ge 5$, $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}}$, $x = x_0 \cdot \ldots \cdot x_r$, $y = y_0 \cdot \ldots \cdot y_r$, $x_i = x_i' x_i''$, and $y_i = y_i' y_i''$ as above, and $|x_i| = |y_i| \ge 2$ for all $i \in [0, r]$. We shall use **A1** and Lemma 3.8.2 again and again without mentioning this explicitly. Of course, we may assume that there is no $a \in \mathcal{A}(\mathcal{B})$ such that $a \mid_{\mathcal{B}} x$ and $a \mid_{\mathcal{B}} y$, since then there is, trivially, a monotone $\mathcal{R}$-chain concatenating x and y. For now, assume $|x|_{\mathcal{B}} \ge 4$; the remaining case, where $|x|_{\mathcal{B}} = 3$, will be studied at the end of the proof after Case 3.

Case 1. There is some $i \in [1, r]$ such that $|x_i'| \ge 1$ and $|y_i'| \ge 1$.

Case 1.1. There is some $i \in [1, r]$ such that $|x_i|' \ge 2$ and $|y_i'| \ge 1$.
Let $a_1, a_2, b \in \mathcal{A}(D_i)$ be such that $a_1 a_2 \mid x_i'$ and $b \mid y_i'$. Then there is some $b' \in \mathcal{A}(D_i)$ such that $a_1 a_2 = b b'$. Thus $\iota(b') = \mathbf{0}$, and if $x^* \in \mathsf{Z}(\mathcal{B})$ is such that $x = a_1 a_2 x^*$, then $x, b b' x^*, y$ is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 1.2. There is some $i \in [1, r]$ such that $|x_i'| = 1$ and $|y_i'| \ge 1$.
Then $x_i' \in \mathcal{A}(\mathcal{B})$. Let $a, b \in \mathcal{A}(D_i)$ be such that $a \mid x_i''$ and $b \mid y_i'$. Let $u \in \mathcal{A}(F)$ be such that $au \in \mathcal{A}(\mathcal{B})$ and $au \mid_{\mathcal{B}} x$. Since $x_i' \in \mathcal{A}(D_i)$, we obtain $u \notin \mathcal{A}(D_i)$. Let $b' \in \mathcal{A}(D_i)$ be such that $x_i' a = b b'$, whence $\iota(b') = g$ and $b' u \in \mathcal{A}(\mathcal{B})$. If $x = x_i'(au)x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, then $|x^*|_{\mathcal{B}} \ge 1$, and $x, b(b'u)x^*, y$ is a monotone $\mathcal{R}$-chain concatenating x and y.

Reduction 1. By Case 1, we may now assume that, for all $i \in [1, r]$, either $|x_i'| = 0$ or $|y_i'| = 0$. In particular, if $|x_i'| \ge 1$, then $|y_i'| = 0$, and therefore $|x_i'| \ge 2$, since $|x_i'| \equiv |y_i'| \mod 2$. Similarly, if $|y_i'| \ge 1$, then $|y_i'| \ge 2$.

Case 2. There is some $i \in [1, r]$ such that $|y_i'| \ge 1$.
In this case, $|x_i'| = 0$ by Reduction 1, hence $|y_i'| \ge 2$ and $|x_i''| \ge 2$. Let $b \in \mathcal{A}(D_i)$ be such that $b \mid y_i'$. Now we must distinguish a few more cases based on $|x|_{\mathcal{B}}$ and $|y|_{\mathcal{B}}$.

Case 2.1. $|x|_{\mathcal{B}} = |y|_{\mathcal{B}}$.
Note that in this case $|x|_{\mathcal{B}} = |y|_{\mathcal{B}} \ge 5$. We assert that there is some $j \in [1, r] \setminus \{i\}$ such that $|y_j'| < |x_j'|$. Indeed, if $|y_j'| \ge |x_j'|$ for all $j \in [1, r] \setminus \{i\}$, then

$$\sum_{\nu=1}^{r} |x_\nu'| \le \sum_{\substack{\nu=1 \\ \nu \ne i}}^{r} |y_\nu'| < \sum_{\nu=1}^{r} |y_\nu'|,$$

and therefore $|x|_{\mathcal{B}} < |y|_{\mathcal{B}}$, a contradiction. By Reduction 1, we obtain $|y_j'| = 0$. Hence $|y_j''| \ge 2$, and $|x_j'| \ge 2$. We write x in the form

$$x = (a_1 u_1) \cdot \ldots \cdot (a_k u_k)(a_{k+1} u_1^*) \cdot \ldots \cdot (a_{k+t} u_t^*)(e_1 u_{k+1}) \cdot \ldots \cdot (e_s u_{k+s}) \tilde{x},$$

where $k, s, t \in \mathbb{N}_0$, $x_i'' = a_1 \cdot \ldots \cdot a_{k+t}$, $x_j'' = u_1 \cdot \ldots \cdot u_{k+s}$, $u_1^*, \ldots, u_t^*, e_1, \ldots, e_s \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $k + t \ge 2$, and

$$\tilde{x} = (e_1 \cdot \ldots \cdot e_s)^{-1} \prod_{\substack{\nu=1 \\ \nu \ne i}}^{r} x_\nu' \prod_{\substack{\nu=1 \\ \nu \ne i,j}}^{r} x_\nu'' \in \mathsf{Z}(\mathcal{B}).$$

Let $c_1, c_2, d_1 \in \mathcal{A}(D_j)$ be such that $c_1 c_2 \mid x_j'$, $d_1 \mid y_j''$, and choose $d_2 \in \mathcal{A}(D_j)$ such that $c_1 c_2 = d_1 d_2$, whence $\iota(d_2) = g$.

Case 2.1a. $t \ge 2$.
Choose some $b' \in \mathcal{A}(D_i)$ such that $a_{k+1} a_{k+2} = b b'$. Then $\iota(b') = \mathbf{0}$, $d_1 u_1^*, d_2 u_2^* \in \mathcal{A}(\mathcal{B})$, and we set

$x = (a_{k+1}u_1^*)(a_{k+2}u_2^*)c_1c_2x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Now x, $bb'(d_1u_1^*)(d_2u_2^*)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1b. $t = 1$.

Note that $|x_i''| = k+1 \ge 2$ implies $k \ge 1$. Assume first that there is some $v \in \mathcal{A}(\mathcal{B})$ such that $|v| = 2$ and $v \mid \tilde{x}$, say $v = v'v''$, where v', $v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$ and $\iota(v') = \iota(v'') = g$. Then it follows that a_1v', $u_1v'' \in \mathcal{A}(\mathcal{B})$, and we set $x = (a_1u_1)(a_{k+1}v_1)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. We set $x' = (a_1v')(a_{k+1}v_1)(u_1v'')x^*$. Then we find $x' \in \mathsf{Z}(\mathcal{B})$, $(x, x') \in \sim_{\mathcal{B}}$, and $x \approx_{\mathrm{eq}} x'$. Hence, $(x', y) \in \sim_{\mathcal{B}}$, $x' \approx_{\mathrm{eq}} y$ by Case 2.1a, and therefore $x \approx_{\mathrm{eq}} y$.

Now we set $u_1^* = u$, and we let $m \in [0, r] \setminus \{i, j\}$ be such that $u \in \mathcal{A}(D_m)$. We write x in the form

$$x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(a_{k+1}u)(e_1u_{k+1}) \cdot \ldots \cdot (e_su_{k+s}) \prod_{\substack{\nu=1 \\ \nu \ne i}}^{r} x_\nu', \quad \text{where } s+1 = \sum_{\substack{\nu=0 \\ \nu \ne i,j}}^{r} |x_\nu''|.$$

We may assume that $|x_n'| = 0$ for all $n \in [1, r] \setminus \{m, j\}$. Indeed, let $n \in [1, r] \setminus \{m, j\}$ be such that $|x_n'| \ge 1$. Then $|x_n'| \ge 2$, $|y_n'| = 0$, and $|y_n''| \ge 2$. Let v_1, v_2, $w_1 \in \mathcal{A}(D_n)$ be such that $v_1v_2 \mid x_n'$ and $w_1 \mid y_n''$, and choose $b_1 \in \mathcal{A}(D_i)$ and $w_2 \in \mathcal{A}(D_n)$ such that $a_1a_{k+1} = bb_1$ and $v_1v_2 = w_1w_2$. Then $\iota(b_1) = \mathbf{0}$, $\iota(w_2) = g$, u_1w_1, $uw_2 \in \mathcal{A}(\mathcal{B})$, and if $x = (a_1u_1)(a_{k+1}u)v_1v_2x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, then $|x^*|_{\mathcal{B}} \ge 1$, and x, $bb_1(u_1w_1)(uw_2)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Thus suppose that $|x_n'| = 0$ for all $n \in [1, r] \setminus \{m, j\}$, and consequently

$$x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(a_{k+1}u)(e_1u_{k+1}) \cdot \ldots \cdot (e_su_{k+s})x_j'x_m'.$$

We assert that there exist v_1, v_2, $v_3 \in \mathcal{A}(D_m)$ and w_1, w_2, $w_3 \in \mathcal{A}(D_j)$ such that $v_1v_2v_3 \mid y_m$, $w_1w_2w_3 \mid y_j$, $\iota(v_\nu) = \iota(w_\nu) = g$, $v_\nu w_\nu \in \mathcal{A}(\mathcal{B})$ and $v_\nu w_\nu \mid_{\mathcal{B}} y$ for all $\nu \in [1, 3]$. Indeed, observe that

$$|y_i''| = |y_i| - |y_i'| \le |y_i| - 2 = |x_i| - 2 = |x_i''| - 2 = k - 1,$$
$$|y_j''| = |y_j| = |x_j'| + |x_j''| \ge 2 + |x_j''| = k + s + 2,$$

and set $y_j'' = y_{j,1} \cdot \ldots \cdot y_{j,\mu}$, where $\mu = |y_j''|$, and, for all $\alpha \in [1, \mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1, \mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(B)$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. Since $|x_j'| \ge 1$, it follows that $u_{j,\alpha} \notin \mathcal{A}(D_j)$ for all $\alpha \in [1, \mu]$. For $\nu \in [0, r] \setminus \{j\}$, we set $N_\nu = |\{\alpha \in [1, \mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu\}|$, and we obtain

$$\mu = \sum_{\substack{\nu=0 \\ \nu \ne j}}^{r} N_\nu = N_m + N_i + \sum_{\substack{\nu=0 \\ \nu \ne i,j,m}}^{r} N_\nu \le N_m + |y_i''| + \sum_{\substack{\nu=0 \\ \nu \ne i,j,m}}^{r} |y_\nu|.$$

Since $|y_\nu| = |x_\nu| = |x_\nu''|$ for all $\nu \in [0, r] \setminus \{i, j, m\}$ and $|x_m''| \ge 1$, it follows that

$$k + s + 2 \le \mu \le N_m + k - 1 + \sum_{\substack{\nu=0 \\ \nu \ne i,j,m}}^{r} |x_\nu''| \le N_m + k - 1 + \sum_{\substack{\nu=0 \\ \nu \ne i,j}}^{r} |x_\nu''| - |x_m''| = N_m + k + s - 1,$$

and therefore $N_m \ge 3$, which implies the existence of v_1, v_2, v_3 and w_1, w_2, w_3 as asserted. In particular, it follows that $|x_m| = |y_m| \ge |y_m''| \ge 3$ and $|x_j| = |y_j| \ge |y_j''| \ge 3$. Let $u_1' \in \mathcal{A}(D_j)$ be such that $u_1u_{k+1} = u_1'w_1$. Then $\iota(u_1') = g$ and $a_1u_1' \in \mathcal{A}(\mathcal{B})$.

Case 2.1b′. $s \ge 1$.

We assume first that $|x_m'| \ge 1$. Let $u' \in \mathcal{A}(D_m)$ be such that $u' \mid x_m'$. Then there exists some $v \in \mathcal{A}(D_m)$ such that $uu' = v_1v$. Hence $\iota(v) = \mathbf{0}$, and $x = (a_1u_1)(a_{k+1}u)(e_1u_{k+1})u'x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Since $a_1u_1' \in \mathcal{A}(\mathcal{B})$ and $a_{k+1}e_1 \in \mathcal{A}(\mathcal{B})$, we conclude that x, $(a_1u_1')(a_{k+1}e_1)(v_1w_1)vx^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Assume now that $|x_m'| = 0$. Then $|x_m''| = |x_m| \ge 3$, and (after renumbering if necessary) we may assume that $e_1 \in \mathcal{A}(D_m)$. Let $v \in \mathcal{A}(D_m)$ be such that $ue_1 = v_1v$. Then $\iota(v) = g$, $a_{k+1}v \in \mathcal{A}(\mathcal{B})$ and $x = (a_1u_1)(a_{k+1}u)(e_1u_{k+1})x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Hence x, $(a_1u_1')(a_{k+1}v)(v_1w_1)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1b″. $s = 0$.

We assert that $k \ge 2$. Indeed, assuming to the contrary that $k = 1$, then $x_i = x_i'' = a_1a_2$, $x_j'' = u_1$, $u_m'' = u$, $3 \le |x_j| = 1 + |x_j'|$, $3 \le |x_m| = 1 + |x_m'|$, hence $|x_j'| \ge 2$, $|x_m'| \ge 2$, and therefore $|y_j'| = |y_m'| = 0$. Hence

$$\sum_{\nu=1}^{r} |y_\nu'| = |y_i'| \le |y_i| = 2 \quad \text{and} \quad \sum_{\nu=1}^{r} |x_\nu'| = |x_j'| + |x_m'| \ge 4,$$

a contradiction to $|x|_{\mathcal{B}} = |y|_{\mathcal{B}}$.

As $k \ge 2$, it follows that $u_2 \in \mathcal{A}(D_j)$, hence $u_2v_1 \in \mathcal{A}(\mathcal{B})$, and we choose $b_2 \in \mathcal{A}(D_i)$ such that $a_1a_2 = bb_2$, whence $\iota(b_2) = \mathbf{0}$. Since $3 \le |x_m| = 1 + |x_m'|$, we get $|x_m'| \ge 2$, and there exist v_1', $v_2' \in \mathcal{A}(D_m)$ such that $v_1'v_2' \mid x_m'$. Let $v \in \mathcal{A}(D_m)$ be such that $v_1'v_2' = v_1v$. Then $\iota(v) = g$ and $u_1v \in \mathcal{A}(\mathcal{B})$.

Assume first that $k \ge 2$, and set $x = (a_1u_1)(a_2u_2)v_1'v_2'x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Then $u_2v_1 \in \mathcal{A}(\mathcal{B})$, and therefore $x,\, bb_2(u_1v)(u_2v_1)x^*,\, y$ is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1c. $t = 0$.

Observe that $|x_i''| = k \ge 2$ and $x = (a_1u_1)\cdot \ldots \cdot (a_ku_k)(e_1u_{k+1})\cdot \ldots \cdot (e_su_{k+s})\,\tilde{x}$. We may assume that there is no $v \in \mathcal{A}(\mathcal{B})$ such that $|v| = 2$ and $v \mid \tilde{x}$. Indeed, if $v \in \mathcal{A}(B)$ is such that $|v| = 2$ and $v \mid \tilde{x}$. Then $v = v'v''$, where $v',\, v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $\iota(v') = \iota(v'') = g$, and $a_2v',\, u_2v'' \in \mathcal{A}(\mathcal{B})$. We set $x = (a_1u_1)(a_2u_2)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, and $x' = (a_1u_1)(a_2v')(u_2v'')x^*$. Then it follows that $x' \in \mathsf{Z}(\mathcal{B})$, $(x, x') \in \sim_{\mathcal{B}}$ and $x \approx_{\mathrm{eq}} x'$. Hence $(x', y) \in \sim_{\mathcal{B}}$, $x' \approx_{\mathrm{eq}} y$ by Case 2.1b, and therefore $x \approx_{\mathrm{eq}} y$. Next we prove that there is some $n \in [1, r] \setminus \{j\}$ such that $|x_n'| \ge 1$. Assume the contrary. Then $x = (a_1u_1)\cdot \ldots \cdot (a_ku_k)(e_1u_{k+1})\cdot \ldots \cdot (e_su_{k+s})x_j'$, $x_i = x_i'' = a_1 \cdot \ldots \cdot a_k$, $x_j'' = u_1 \cdot \ldots \cdot u_{k+s}$, and

$$e_1 \cdot \ldots \cdot e_s = \prod_{\substack{\nu=0\\ \nu \ne i,j}}^{r} x_\nu\,.$$

Moreover, we obtain $|y_i''| = |y_i| - |y_i'| \le |x_i| - 2 = |x_i''| - 2 = k-2$ and $|y_j''| = |y_j| = |x_j'| + |x_j''| \ge 2 + k + s$. We set $y_j'' = y_{j,1} \cdot \ldots \cdot y_{j,\mu}$, where $\mu = |y_j''|$, and, for all $\alpha \in [1, \mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1, \mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(B)$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. Since $|x_j'| \ge 1$, it follows that $u_{j,\alpha} \notin \mathcal{A}(D_j)$ for all $\alpha \in [1, \mu]$. For $\nu \in [0, r] \setminus \{j\}$, we set $N_\nu = |\{\alpha \in [1, \mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu\}|$, and we obtain

$$2 + k + s \;\le\; \mu \;=\; \sum_{\substack{\nu=0\\ \nu\ne j}}^{r} N_\nu \;\le\; \sum_{\substack{\nu=0\\ \nu\ne j}}^{r} |y_\nu''| \;\le\; |y_i''| + \sum_{\substack{\nu=0\\ \nu\ne i,j}}^{r} |y_\nu| \;=\; |y_i''| + \sum_{\substack{\nu=0\\ \nu\ne i,j}}^{r} |x_\nu| \;\le\; k - 2 + s\,,$$

a contradiction.

Thus now let $n \in [1, r] \setminus \{j\}$ be such that $|x_n'| \ge 1$. Then $|x_n'| \ge 2$, $|y_n'| = 0$ and $|y_n''| \ge 2$. Let $v_1,\, v_2,\, w_1 \in \mathcal{A}(D_n)$ be such that $v_1v_2 \mid x_n'$, $w_1 \mid y_n''$, and choose some $x_2 \in \mathcal{A}(D_n)$ such that $v_1v_2 = w_1w_2$. Then $x = (a_1u_1)(a_2u_2)v_1v_2x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*| \ge 1$. Let $b_2 \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb_2$, whence $\iota(b_2) = \mathbf{0}$. Then $x,\, bb_2(u_1w_1)(u_2w_2)x^*,\, y$ is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.2. $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}} + 1$, and we are in the following special situation.

S1 There exist $a_1,\, a_2 \in \mathcal{A}(D_i)$ and $u_1,\, u_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that $a_1u_1,\, a_2u_2,\, u_1u_2 \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1)(a_2u_2) \mid_{\mathcal{B}} x$.

We set $x = (a_1u_1)(a_2u_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x|_{\mathcal{B}} \ge 1$, and we let $b' \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb'$, whence $\iota(b') = 0$. Then $x,\, bb'(u_1u_2)x^*,\, y$ is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.3. $|y|_{\mathcal{B}} = |x|_{\mathcal{B}} + 1$, and we are not in the special situation **S1**.

We set $x_i'' = a_1 \cdot \ldots \cdot a_k$, where $a_1, \ldots, a_k \in \mathcal{A}(D_i)$ and $k \ge 2$. For $\nu \in [1, k]$, let $u_\nu \in \mathcal{A}(F)$ be such that $a_\nu u_\nu \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1)\cdot \ldots \cdot (a_ku_k) \mid_{\mathcal{B}} x$. Since $|y_i'| \ge 1$ and we are not in the special situation **S1**, there exists some $j \in [1, r] \setminus \{i\}$ such that $u_1 \cdot \ldots \cdot u_k \mid x_j''$. Suppose that $x_j'' = u_1 \cdot \ldots \cdot u_{k+s}$, where $s \in \mathbb{N}_0$, and let $c_1, \ldots, c_s \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$ be such that $x = (a_1u_1)\cdot \ldots \cdot (a_ku_k)(c_1u_{k+1})\cdot \ldots \cdot (c_su_{k+s})\tilde{x}$ for some $\tilde{x} \in \mathsf{Z}(\mathcal{B})$.

We may assume that there is no $v \in \mathcal{A}(\mathcal{B})$ such that $|v| = 2$ and $v \mid \tilde{x}$. Indeed, suppose that $v \in \mathcal{A}(B)$ is such that $|v| = 2$ and $v \mid \tilde{x}$. Then $v = v'v''$, where $v',\, v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $\iota(v') = \iota(v'') = g$, and $a_2v',\, u_2v'' \in \mathcal{A}(\mathcal{B})$. We set $x = (a_1u_1)(a_2u_2)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, and $x' = (a_1u_1)(a_2v')(u_2v'')x^*$. Then it follows that $x' \in \mathsf{Z}(\mathcal{B})$, $(x, x') \in \sim_{\mathcal{B}}$ and $x \approx_{\mathrm{eq}} x'$. Hence $(x', y) \in \sim_{\mathcal{B}}$, and, by Case 2.2, there is a monotone $\mathcal{R}$-chain concatenating x' and y, and therefore there is a monotone $\mathcal{R}$-chain concatenating x and y.

Hence x is of the form

$$x = (a_1u_1)\cdot \ldots \cdot (a_ku_k)(c_1u_{k+1})\cdot \ldots \cdot (c_su_{k+s})x_1' \cdot \ldots \cdot x_r',$$

and we assert that there exists some $m \in [1, r] \setminus \{j\}$ such that $|x_m'| \ge 2$. Indeed, if we assume to the contrary that $|x_m'| = 0$ for all $m \in [1, r] \setminus \{j\}$, then we obtain $|x| = 2(k+s) + |x_j'|$, and since $|y|_{\mathcal{B}} = |x|_{\mathcal{B}} + 1$, it follows that

$$\sum_{\nu=0}^{r} |y_\nu'| = \sum_{\nu=0}^{r} |x_\nu'| + 2 = |x_j'| + 2.$$

If $|y_j'| \ge 1$, then we find $|x_j'| = 0$ and $|y_j'| \ge 2$, and therefore

$$4 \le |y_j'| + |y_i'| \le \sum_{\nu=0}^{r} |y_\nu'| = 2,$$

a contradiction. Hence it follows that $|y'_j| = 0$, and then $|y''_j| = |y_j| = |x_j| = k + s + |x'_j|$. Now we find

$$\sum_{\substack{\nu=0\\ \nu\neq j}}^{r} |y''_\nu| \le |y| - |y''_j| - |y'_i| \le |x| - (k+s+|x'_j|) - 2 = k+s-2 \le |y''_j| - 2.$$

We set $y''_j = y_{j_1} \cdot \ldots \cdot y_{j,\mu}$, where $\mu = |y''_j|$ and, for all $\alpha \in [1,\mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1,\mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(\mathcal{B})$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. For $\nu \in [0,r]$, we set $N_\nu = \#\{\alpha \in [1,\mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu)\}$, and we obtain

$$0 \le \sum_{\substack{\nu=0\\ \nu\neq j}}^{r} |y''_\nu| \le |y''_j| - 2 = \sum_{\nu=0}^{r} N_\nu - 2,$$

and therefore $N_j \ge 2$. Hence, there exist w_1, $w_2 \in \mathcal{A}(D_j)$ such that $\iota(w_1) = \iota(w_2) = g$ and $w_1w_2 \in \mathcal{A}(\mathcal{B})$. On the other hand, $u_1u_2 \notin \mathcal{A}(\mathcal{B})$, since we are not in the special situation **S1**, and therefore $u_1u_2 = u'_1u'_2$, where u'_1, $u'_2 \in \mathcal{A}(D_j)$ and $\iota(u'_1) = \iota(u'_2) = \mathbf{0}$. Hence the existence of $w_1w_2 \in \mathcal{A}(\mathcal{B})$ contradicts **A1**.
Let now $m \in [1,r] \setminus \{j\}$ be such that $|x'_m| \ge 2$ and let $b' \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb'$. By Reduction 1, we obtain $|y'_m| = 0$, hence $|y''_m| \ge 2$, there exist v', $v'' \in \mathcal{A}(D_m)$ such that $v'v'' \mid x'_m$, and there exists some $u' \in \mathcal{A}(D_m)$ such that $u' \mid y''_m$. Let $u'' \in \mathcal{A}(D_m)$ be such that $v'v'' = u'u''$, whence $\iota(u'') = g$, and set $x = (a_1u_1)(a_2u_2)v'v''x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$. If $|x|_{\mathcal{B}} = 4$, then $x = (a_1u_1)(a_2u_2)v'v''$ and thus $y = y'_iy'_jy''_m$, where $|y'_i| = |y'_j| = |y''_m| = 2$, and thus there is a pure atom in m dividing y. Since v', $v'' \in \mathcal{A}(D_m)$ and $\iota(v') = \iota(v'') = \mathbf{0}$, this contradicts **A1**. Now we may assume $|x|_{\mathcal{B}} \ge 5$. Then we have $|x^*|_{\mathcal{B}} \ge 1$ and it follows that $u_1u', u_2u'' \in \mathcal{A}(\mathcal{B})$, and x, $bb'(u_1u')(u_2u'')x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.4. $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}} + 2$, and we are not in the special situation **S1**.
Let a_1, $a_2 \in \mathcal{A}(D_i)$ be such that $a_1a_2 \mid x''_i$. Since $|y'_i| > 0$, there are u_1, $u_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that a_1u_1, $a_2u_2 \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1)(a_2u_2) \mid_{\mathcal{B}} x$. We set $x = (a_1u_1)(a_2u_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*| \ge 1$, and $u_1u_2 = v_1v_2$ for some v_1, $v_2 \in \mathcal{A}(D_i)$ such that $\iota(v_1) = \iota(v_2) = \mathbf{0}$. Again we set $a_1a_2 = bb'$, where $b' \in \mathcal{A}(D_i)$ and $\iota(b') = \mathbf{0}$, and then x, $bb'v_1v_2x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y, since $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}} + 2 = |x^*| + 4$.

Reduction 2. By Case 2, we may now assume that $|y'_i| = 0$ for all $i \in [1,r]$, and since $|x|_{\mathcal{B}} \le |y|_{\mathcal{B}}$, this implies that $|x'_i| = 0$. Therefore $|x''_i| \ge 2$ and $|y''_i| \ge 2$ for all $i \in [1,r]$. Since $x''_0 = x_0 = y_0 = y''_0$, we have $x_i = x''_i$ for all $i \in [0,r]$.

Case 3. $x_i = x''_i$, $y_i = y''_i$, and $|x_i| = |y_i| \ge 2$ for all $i \in [0,r]$.

Case 3a. There is some $i \in [0,r]$ such that

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |x_\nu| < |x_i| \quad \left[\text{and thus also } \sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |y_\nu| < |y_i|\right].$$

There exist a_1, a_2, b_1, $b_2 \in \mathcal{A}(D_i)$ such that $a_1a_2 \in \mathcal{A}(\mathcal{B})$, $b_1b_2 \in \mathcal{A}(\mathcal{B})$, $a_1a_2 \mid_{\mathcal{B}} x$, and $b_1b_2 \mid_{\mathcal{B}} y$. Let $b \in \mathcal{A}(D_i)$ be such that $a_1a_2 = b_1b$. Since $5 \le |x|_{\mathcal{B}} \le 2|x''_i| = 2|x_i|$, there exists some $a_w \in \mathcal{A}(D_i)$ such that $a_1a_2a_3 \mid x_i$. Let $c \in \mathcal{A}(F)$ be such that $a_3c \in \mathcal{A}(\mathcal{B})$ and $a_3c \mid_{\mathcal{B}} x$, and let $b_3 \in \mathcal{A}(D_i)$ be such that $ba_3 = b_2b_3$. If $x = (b_1b)(a_3c)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$, then x, $(b_1b_2)(b_3c)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3b. For all $i \in [0,r]$, we have

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |x_\nu| \ge |x_i| \quad \left[\text{and thus also } \sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |y_\nu| \ge |y_i|\right].$$

We shall prove the following reduction step.

R1 We may assume that, for each $i \in [0,r]$, there is no pure atom in i dividing either x or y in $\mathcal{B}$.

Proof of **R1**. Let $\tilde{x} \in \mathsf{Z}(\mathcal{B})$ be such that $(x,\tilde{x}) \in \sim_{\mathcal{B}}$, $x \approx_{\mathrm{eq}} \tilde{x}$, and the number of pure atoms dividing $\tilde{x}$ is minimal. Assume there is at least one pure atom in $i \in [0,r]$ dividing $\tilde{x}$, say $a_1a_2 \in \mathcal{A}(\mathcal{B})$ with a_1, $a_2 \in \mathcal{A}(D_i)$ and $a_1a_2 \mid_{\mathcal{B}} \tilde{x}$. Now we find

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |\tilde{x}_\nu| \ge |\tilde{x}_i| - 2,$$

and thus there are c_1, $c_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ with $c_1c_2 \in \mathcal{A}(\mathcal{B})$ and $c_1c_2 \mid_{\mathcal{B}} \tilde{x}$. If $\tilde{x} = (a_1a_2)(c_1c_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$, then we set $x' = (a_1c_1)(a_2c_2)x^*$. Now we find $(\tilde{x},x') \in \sim_{\mathcal{B}}$ and $\tilde{x} \approx_{\mathrm{eq}} x'$, and

thus $x \approx_{\mathrm{eq}} x'$. Since there is one pure atom less dividing x' than $\tilde{x}$, this is a contradiction.
The same argument applies to y. Therefore there exist $\tilde{x}$, $\tilde{y} \in \mathsf{Z}(\mathcal{B})$ both not divisible by any pure atom such that $(x, \tilde{x})$, $(y, \tilde{y}) \in \sim_{\mathcal{B}}$, $x \approx_{\mathrm{eq}} \tilde{x}$, and $y \approx_{\mathrm{eq}} \tilde{y}$. Hence it follows that $(\tilde{x}, \tilde{y}) \in \sim_{\mathcal{B}}$ and if $\tilde{x} \approx_{\mathrm{eq}} \tilde{y}$, then $x \approx_{\mathrm{eq}} y$. ☐

Next we prove the following reduction step.

R2 We may assume that, for each $i \in [0, r]$, $x_i = y_i$.

Proof of **R2**. Trivially, we have $x_0 = y_0$. Now let $i \in [1, r]$. We assert that there is some $\tilde{x} \in \mathsf{Z}(\mathcal{B})$ such that $(x, \tilde{x}) \in \sim_{\mathcal{B}}$, $x \approx_{\mathrm{eq}} \tilde{x}$, and $z = \gcd(\tilde{x}_i, y_i)$ (in F) is maximal. Now assume that $\tilde{x}_i = z\tilde{z}$ and $y_i = z\tilde{z}'$, where $\tilde{z}$, $\tilde{z}' \in \mathsf{Z}(D_i)$ and $|\tilde{z}| = |\tilde{z}'| \geq 1$. If $|\tilde{z}| = |\tilde{z}'| = 1$, then there are some v, $v' \in \mathcal{A}(D_i)$ such that $\tilde{z} = v$ and $\tilde{z}' = v'$. Now we find

$$\pi_i(z)v = \pi_i(z\tilde{z}) = \pi_i(\tilde{x}_i) = \pi_i(y_i) = \pi_i(z\tilde{z}') = \pi_i(z)v',$$

and thus $v = v'$. But then $\gcd(\tilde{x}_i, y_i) = vz$, a contradiction. If $|\tilde{z}| = |\tilde{z}'| \geq 2$, then there are a_1, a_2, $b \in \mathcal{A}(D_i)$ with $a_1a_2 \mid \tilde{x}_i$ and $b \mid y_i$. By **R1**, there are c_1, $c_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that a_1c_1, $a_2c_2 \in \mathcal{A}(\mathcal{B})$ and a_1c_1, $a_2c_2 \mid_{\mathcal{B}} x$. There is some $b' \in \mathcal{A}(D_i)$ such that $a_1a_2 = b'b$. If $\tilde{x} = (a_1c_1)(a_2c_2)x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \geq 1$, then we set $\bar{x} = (bc_1)(b'c_2)x^*$ and find $(\tilde{x}, \bar{x}) \in \sim_{\mathcal{B}}$ and $\tilde{x} \approx_{\mathrm{eq}} \bar{x}$, and thus $x \approx_{\mathrm{eq}} \bar{x}$. Since $bz = b\gcd(\tilde{x}_i, y_i) \mid \gcd(\bar{x}_i, y_i)$, this is a contradiction. ☐

Now we fix—again arbitrarily—some $i \in [0, r]$ and choose $a \in \mathcal{A}(D_i)$ such that $a \mid x_i''$. Then $a \mid y_i''$, too. By **R1**, there are c, $d \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that $ac \mid_{\mathcal{B}} x$ and $ad \mid_{\mathcal{B}} y$. Again by **R1**, there are e, $f \in \mathcal{A}(F)$ such that $de \mid_{\mathcal{B}} x$ and $cf \mid_{\mathcal{B}} y$.
Then x and y are of the following forms

$$x = (ac)(de)x^* \quad \text{and} \quad y = (ad)(cf)y^*,$$

where x^*, $y^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} = |y^*|_{\mathcal{B}} \geq 1$.

Case 3.b$'$. $ce \in \mathcal{A}(\mathcal{B})$.
Then x, $(ad)(ce)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3.b$''$ $df \in \mathcal{A}(\mathcal{B})$.
Then x, $(ac)(df)y^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3.b$'''$ We are neither in Case 3.b$'$ nor in Case 3.b$''$, and thus there are j_1, $j_2 \in [0, r] \setminus \{i\}$ with $j_1 \neq j_2$ such that c, $e \in \mathcal{A}(D_{j_1})$ and d, $f \in \mathcal{A}(D_{j_2})$. Then $ae, af, cd \in \mathcal{A}(\mathcal{B})$ and hence x, $(ae)(cd)x^*$, $(af)(cd)y^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Now it remains to prove the special case, where $|x|_{\mathcal{B}} = 3$. By the length formulas from the beginning of the proof, we find that $|y|_{\mathcal{B}} \in \{5, 6\}$. If $|y|_{\mathcal{B}} = 5$, then the length formulas imply that that there is some $i \in [1, r]$ such that $|x_i'| = 1$ and $|y_i'| \geq 1$, and thus we are in the situation of Case 1.2. When we inspect the monotone $\mathcal{R}$-chain constructed there, we find that the same monotone $\mathcal{R}$-chain concatenating x and y exists in our situation. If $|y|_{\mathcal{B}} = 6$, then we find that $|x_i'| = 0$ and $|y_i''| = 0$ for all $i \in [1, r]$. Since $6 = |y|_{\mathcal{B}} \geq |x|_{\mathcal{B}} + 2 = 5$, we are either in Case 2.2 or in Case 2.4. Again we inspect the monotone $\mathcal{R}$-chains constructed there and we find that the same monotone $\mathcal{R}$-chains concatenating x and y exist in our special situation.
Now it remains to show part 2. By [21, Lemma 3.4], we have

$$\begin{aligned}\mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}}), &\text{ there is no monotone } \mathcal{R}\text{-chain from } x \\ &\text{to } y, \text{ and either } |x| = |y| \text{ or } |x|, |y| \in \mathsf{L}(\pi_{\mathcal{B}}(x)) \text{ are adjacent}\}.\end{aligned}$$

By part 1, there is a monotone $\mathcal{R}$-chain concatenating x and y for all $(x, y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \geq 5$ and $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}}$. Thus it suffices to consider relations $(x, y) \in \sim_{\mathcal{B}}$ with $|x|_{\mathcal{B}} \leq |y|_{\mathcal{B}} \leq 4$. By definition, we have $\{(x, y) \in \mathcal{A}(\sim_{\mathcal{B}}) \mid |x|_{\mathcal{B}} \leq |y|_{\mathcal{B}}\} \subset \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}})$. Since the shortest possible atom $(x, y) \in \mathcal{A}(\sim_{\mathcal{B}}) \setminus \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}})$ satisfies $|x|_{\mathcal{B}} > |y|_{\mathcal{B}} \geq 2$, we find $|xy|_{\mathcal{B}} \geq 5$. Hence, we may restrict to elements of $\mathcal{A}(\sim_{\mathcal{B}})$, and the assertion follows. ☐

Using Lemma 3.14.4, Lemma 3.15, and Lemma 3.16 above, we can now calculate the catenary degree and the minimum distance (when $|G| = 2$), and in a slightly more special but still interesting situation, we can compute the elasticity, the monotone catenary degree and the tame degree.

Proposition 3.17. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}_0$, $s \in \mathbb{N}_0$, $r + s \geq 1$, and let $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ be reduced half-factorial but not factorial monoids of type $(1, k_i)$ for $i \in [1, r + s]$ with $k_1 = \ldots = k_r = 1$ and $k_{r+1} = \ldots = k_s = 2$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_{r+s}$. Let $H \subset D$ be a saturated submonoid, let $G = \mathsf{q}(D/H)$ be its class group with $|G| = 2$, say $G = \{\mathbf{0}, g\}$, suppose each class in G contains some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_{r+s} \to G$ by $\iota(t) = [t]_{D/H}$.*

Furthermore, set $I = \{i \in [1, r+s] \mid (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2 \neq \emptyset\}$ *and* $J = \{i \in [r+1, r+s] \mid \mathsf{c}(D_i) = 3\}$. *Then*

1. *If* $I = J = \emptyset$*, then* H *is half-factorial and* $\mathsf{c}(H) = 2$.
2. *If* $I = \emptyset$ *and* $J \neq \emptyset$*, then* $\mathsf{c}(H) \in \{2, 3\}$*, and* $\triangle(H) \subset \{1\}$.
3. *If* $\#I = 1$*, then* $\rho(H) \geq \frac{3}{2}$, $\mathsf{c}(H) = 3$*, and* $\triangle(H) = \{1\}$.
4. *If* $\#I \geq 2$*, then* $\rho(H) = 2$, $\mathsf{c}(H) = 4$*, and* $\triangle(H) = \{1, 2\}$.
5. *If* $s = 0$*, then* $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H)$*. Additionally, if* $\#I = 1$*, then* $\rho(H) = \frac{3}{2}$.
6. *If* $s = 0$ *and* $\iota(p_i) = \mathbf{0}$ *for all* $i \in [1, r]$*, then* H *is half-factorial if and only if* $\mathsf{t}(H) = 2$.

In particular, $\min \triangle(H) \leq 1$ *always holds.*

Proof. We set $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_{r+s} \mid \mathbf{0} \nmid S\}$. By Lemma 3.4, H and D are reduced BF-monoids, and $H \subset D$ is a faithfully saturated submonoid. By Lemma 3.4.4, we obtain $\triangle(H) = \triangle(\mathcal{B})$, $\rho(H) = \rho(\mathcal{B})$, $\mathsf{c}(H) = \mathsf{c}(\mathcal{B})$, and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B})$. Lemma 3.14.1 implies $\mathsf{c}(D) \leq 3$, and, by Lemma 3.14.4, we obtain $\mathsf{c}(\mathcal{B}) = \mathsf{c}(H) \leq 4$.
By [20, Proposition 14.1], we obtain $\mathsf{c}(\mathcal{B}) \leq \sup\{|y|_{\mathcal{B}} \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}})$, and since $\mathsf{c}(\mathcal{B}) \leq 4$, it follows that

$$\mathsf{c}(\mathcal{B}) \leq \max\{|y|_{\mathcal{B}} \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}}),\ |x|_{\mathcal{B}} \leq |y|_{\mathcal{B}} \leq 4\};$$

indeed we can replace the supremum with a maximum since we have a bounded set of integers on the right hand side.

If $(x, y) \in \mathcal{A}(\sim_{\mathcal{B}})$, then $(x, y) = (u_1 \cdot \ldots \cdot u_k, v_1 \cdot \ldots \cdot v_l)$, where $k = |x|_{\mathcal{B}}$, $l = |y|_{\mathcal{B}}$, and u_i, $v_j \in \mathcal{A}(\mathcal{B})$ for all $i \in [1, k]$ and $j \in [1, l]$. In this case, we call the atom (x, y) of type (k, l) and describe it by the defining relation $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$ in $\mathcal{B}$. Now the equation from above reads as follows:

$$\mathsf{c}(\mathcal{B}) = \max\{|x|_{\mathcal{B}} \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}}) \text{ is of type } (k, l), \text{ where } 2 \leq k \leq l \leq 4\}.$$

Hence we proceed with a list of defining relations for all atoms of type (k, l), where $2 \leq k \leq l \leq 4$. An atom will be called of character $\mathcal{C} \in [1, 15]$ if it is defined by the relation $(3.1.\mathcal{C})$ in the list below.
Let i, $j \in [1, r+s]$, $i \neq j$. Then

$$g^2(p_i p_j \varepsilon_i \varepsilon_j) = (p_i \varepsilon_i g)(p_j \varepsilon_j g) \tag{3.1}$$

describes an atom of type $(2, 2)$ if and only if $\varepsilon_i \in \mathcal{U}_1(D_i)$, $\varepsilon_j \in \mathcal{U}_1(D_j)$, and $\iota(p_i \varepsilon_i) = \iota(p_j \varepsilon_j) = g$;

$$(p_i p_j \varepsilon_i^{(1)} \varepsilon_j^{(1)})(p_i p_j \varepsilon_i^{(2)} \varepsilon_j^{(2)}) = (p_i^2 \varepsilon_i^{(1)} \varepsilon_i^{(2)})(p_j^2 \varepsilon_j^{(1)} \varepsilon_j^{(2)}) \tag{3.2}$$

describes an atom of type $(2, 2)$ if and only if $\iota(p_i \varepsilon_i^{(1)}) = \iota(p_i \varepsilon_i^{(2)}) = \iota(p_j \varepsilon_j^{(1)}) = \iota(p_j \varepsilon_j^{(2)}) = g$, $\varepsilon_i^{(1)} \varepsilon_i^{(2)} \notin \mathcal{U}_1(D_i)^2_{\iota(p_i)}$, and $\varepsilon_j^{(1)} \varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$;

$$g^2(p_i^2 \varepsilon_1 \varepsilon_2) = (p_i \varepsilon_1 g)(p_i \varepsilon_2 g) \tag{3.3}$$

describes an atom of type $(2, 2)$ if and only if either ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$, $\varepsilon_1 \varepsilon_2 \notin (\mathcal{U}_1(D_i)_g)^2$, and $\iota(p_i) = g$ or ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_g$, $\varepsilon_1 \varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2$, and $\iota(p_i) = \mathbf{0}$;

$$(p_i \varepsilon_1)(p_i \varepsilon_2) = (p_i \eta_1)(p_i \eta_2) \tag{3.4}$$

describes an atom of type $(2, 2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\eta_1) = \iota(\eta_2)$, and $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$;

$$(p_i \varepsilon_1 g)(p_i \varepsilon_2 g) = (p_i \eta_1 g)(p_i \eta_2 g) \tag{3.5}$$

describes an atom of type $(2, 2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i \varepsilon_1) = \iota(p_i \varepsilon_2) = \iota(p_i \eta_1) = \iota(p_i \eta_2) = g$, and $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$;

$$(p_i \varepsilon_1)(p_i \varepsilon_2 g) = (p_i \eta_1)(p_i \eta_2 g) \tag{3.6}$$

describes an atom of type $(2, 2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i) = \iota(\varepsilon_1) = \iota(\eta_1)$, $\iota(p_i \varepsilon_2) = \iota(p_i \eta_2) = g$, and $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$; and

$$(p_i \varepsilon_1 g)(p_i \varepsilon_2 g) = (p_i \eta_1)(p_i \eta_2) g^2\,, \tag{3.7}$$

describes an atom of type $(2, 3)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$, $\iota(p_i \varepsilon_1) = \iota(p_i \varepsilon_2) = g$, and $\iota(p_i \eta_1) = \iota(p_i \eta_2) = \mathbf{0}$. If these conditions are fulfilled, then $\varepsilon_1 \varepsilon_2 \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$ and therefore $i \in I$. Conversely, if $i \in I$, then $\mathcal{U}_1(D_1)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g \neq \emptyset$. If $\iota(p_i) = g$, let ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$ and η_1, $\eta_2 \in \mathcal{U}_1(D_i)_g$ be such that $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$. If $\iota(p_i) = \mathbf{0}$, let ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_g$ and η_1, $\eta_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$ be such that $\varepsilon_1 \varepsilon_2 = \eta_1 \eta_2$. In any case, (3.7) holds.
Now let $i \in I$, $j \in [1, r+s]$, and $i \neq j$. Then

$$(p_i p_j \varepsilon_i^{(1)} \varepsilon_j^{(1)})(p_i p_j \varepsilon_i^{(2)} \varepsilon_j^{(2)}) = (p_i \eta_i^{(1)})(p_i \eta_i^{(2)})(p_j^2 \varepsilon_j^{(1)} \varepsilon_j^{(2)}) \tag{3.8}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_i^{(1)}, \varepsilon_i^{(2)}, \eta_i^{(1)}, \eta_i^{(2)} \in \mathcal{U}_1(D_i)$, $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)$, $\varepsilon_i^{(1)}\varepsilon_i^{(2)} = \eta_i^{(1)}\eta_i^{(2)}$, $\iota(p_i\varepsilon_i^{(1)}) = \iota(p_i\varepsilon_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\varepsilon_j^{(2)}) = g$, $\iota(p_i\eta_i^{(1)}) = \iota(p_i\eta_i^{(2)}) = \mathbf{0}$, and $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$. If these conditions are fulfilled, then $\varepsilon_i^{(1)}\varepsilon_i^{(2)} \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$ and therefore $i \in I$. However, if $i \in I$, then a relation (3.8) need not hold, since we cannot guarantee that there exist $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)$ such that $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$.
Now let i, $j \in I$ and $i \neq j$. Then

$$(p_ip_j\varepsilon_i^{(1)}\varepsilon_j^{(1)})(p_ip_j\varepsilon_i^{(2)}\varepsilon_j^{(2)}) = (p_i\eta_i^{(1)})(p_i\eta_i^{(2)})(p_j\eta_j^{(1)})(p_j\eta_j^{(2)}) \tag{3.9}$$

describes an atom of type $(2,4)$ if and only if $\varepsilon_i^{(1)}$, $\varepsilon_i^{(2)}$, $\eta_i^{(1)}$, $\eta_i^{(2)} \in \mathcal{U}_1(D_i)$, $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)}$, $\eta_j^{(1)}$, $\eta_j^{(2)} \in \mathcal{U}_1(D_j)$, $\varepsilon_i^{(1)}\varepsilon_i^{(2)} = \eta_i^{(1)}\eta_i^{(2)}$, $\varepsilon_j^{(1)}\varepsilon_j^{(2)} = \eta_j^{(1)}\eta_j^{(2)}$, $\iota(p_i\varepsilon_i^{(1)}) = \iota(p_i\varepsilon_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\varepsilon_j^{(2)}) = g$, and $\iota(p_i\eta_i^{(1)}) = \iota(p_i\eta_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\eta_j^{(2)}) = \mathbf{0}$. If these conditions are fulfilled, then $\varepsilon_i^{(1)}\varepsilon_i^{(2)} \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$ and $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_j)^2_g$, and therefore i, $j \in I$. Conversely, if i, $j \in I$, then a relation (3.9) holds (see the arguments for (3.7)). Let $i \in J$, ε_1, ε_2, ε_3, η_1, η_2, $\eta_3 \in \mathcal{U}_1(D_i)$, and $\mathsf{c}_{D_i}((p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3), (p_i\eta_1)(p_i\eta_2)(p_i\eta_3)) = 3$. Then

$$(p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3) \tag{3.10}$$

describes an atom of type $(3,3)$ if and only if $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\varepsilon_3) = \iota(\eta_1) = \iota(\eta_2) = \iota(\eta_3)$;

$$(p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3g) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3g) \tag{3.11}$$

describes an atom of type $(3,3)$ if and only if $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\eta_1) = \iota(\eta_2)$ and $\iota(p_i\varepsilon_3) = \iota(p_i\eta_3) = g$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3) \tag{3.12}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = g$, and $\iota(p_i) = \iota(\varepsilon_1) = \iota(\eta_1) = \iota(\eta_2) = \iota(\eta_3)$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3g) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3g) \tag{3.13}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\varepsilon_3) = \iota(p_i\eta_3) = g$, and $\iota(p_i) = \iota(\eta_1) = \iota(\eta_2)$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3) = (p_i^2\eta_1\eta_2)(p_i\eta_3)\,, \tag{3.14}$$

describes an atom of type $(2,2)$ if and only if $\varepsilon_1\varepsilon_2$, $\eta_1\eta_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\eta_1) = \iota(p_i\eta_2) = g$, and $\iota(p_i) = \iota(\varepsilon_3) = \iota(\eta_3)$; and

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3g) = (p_i^2\eta_1\eta_2)(p_i\eta_3g) \tag{3.15}$$

describes an atom of type $(2,2)$ if and only if $\varepsilon_1\varepsilon_2$, $\eta_1\eta_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, and $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\varepsilon_3) = \iota(p_i\eta_1) = \iota(p_i\eta_2) = \iota(p_i\eta_3) = g$.
Now we can do the actual proof.

1. If $I = J = \emptyset$, then only atoms of characters $[1,6]$ exist, and they are all of type $(2,2)$. Hence, we obtain $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}) = 2$, and thus H is half-factorial.
2. If $I = \emptyset$ and $J \neq \emptyset$, then there are atoms of characters $[1,6] \cup [10,15]$, and they are of types $(2,2)$, $(2,3)$, and $(3,3)$. Hence, it follows that $\mathsf{c}(H) \in \{2,3\}$ and $\triangle(H) \subset \{1\}$.
3. If $\#I = 1$, then atoms of characters $[1,7]$ exist, and atoms of characters $\{8\} \cup [10,15]$ might exist. The atoms of characters $[1,7]$ are of types $(2,2)$ and $(2,3)$, and the atoms of characters $\{8\} \cup [10,15]$ are of types $(2,3)$, $(3,3)$, and $(2,2)$. Thus we have $\rho(H) \geq \frac{3}{2}$ and $\mathsf{c}(H) = 3$, and therefore $\triangle(H) = \{1\}$ by [12, Theorem 1.6.3].
4. If $\#I \geq 2$, then atoms of characters $[1,7] \cup \{9\}$ exist and possibly also atoms of characters $\{8\} \cup [10,15]$ exist, and they are of types $(2,2)$, $(2,3)$, and $(2,4)$. Thus we find $\mathsf{c}(H) = 4$, $\{1,2\} \subset \triangle(H)$, and $\rho(H) \geq 2$. Since $\rho(H) \leq 2$ by Lemma 3.14.4, we obtain the equality $\rho(H) = 2$ and, by [12, Theorem 1.6.3], we find $\triangle(H) = \{1,2\}$.
5. Let $s = 0$. If $I = \emptyset$, then H is half-factorial by part 1, and thus $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H)$ by [21, Lemma 4.4.1].

   If $\#I = 1$, then atoms of characters $[1,7]$ exist, and atoms of character 8 might exist. The atoms of characters $[1,7]$ are of types $(2,2)$ and $(2,3)$, and the atoms of character 8 are also of type $(2,3)$. By Lemma 3.16.2, we have $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \leq 3$. By part 3, we find $3 = \mathsf{c}(H) \leq \mathsf{c}_{\mathrm{mon}}(H)$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = 3$.

   It remains to show that $\rho(H) = \rho(\mathcal{B}) = \frac{3}{2}$. By part 3, we have $\rho(H) \geq \frac{3}{2}$. Thus it suffices to show that $\rho(H) \leq \frac{3}{2}$. Now let $(x,y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}}$. Then there is a monotone 3-chain concatenating x and y, say $x = z_0, z_1, \ldots, z_n = y$ with $z_1, \ldots, z_n \in \mathsf{Z}(\pi_{\mathcal{B}}(x))$ and $n \in \mathbb{N}$. Whenever $|z_{i-1}|_{\mathcal{B}} < |z_i|_{\mathcal{B}}$ for some $i \in [1,n]$, then $\mathsf{d}(z_{i-1}, z_i) = 3$ and there is an atom $(z'_{i-1}, z'_i) \in \mathcal{A}(\sim_H)$ of character 7 or 8 such that $z_{i-1} = d_i z'_{i-1}$ and $z_i = d_i z'_i$, where $d_i = \gcd(z_{i-1}, z_i)$. Since atoms of both characters replace two very special atoms in $\mathcal{A}(\mathcal{B})$ (on the left side) by three different atoms (on the right side) and there is no atom of character $x \in [1,6]$, which generates the first special atoms, there are at most $\frac{1}{2}|x|_{\mathcal{B}}$ such steps, and thus $|y|_{\mathcal{B}} \le \frac{3}{2}|x|_{\mathcal{B}}$, which proves $\rho(H) \le \frac{3}{2}$.

If $\#I \ge 2$, then atoms of characters $[1,7] \cup \{9\}$ exist, and possibly also atoms of character 8 exist. The atoms of characters $x \in [1,7] \cup \{9\}$ are of types $(2,2)$, $(2,3)$, and $(2,4)$, and the atoms of character 8 are of type $(2,3)$. By Lemma 3.16.2, we have $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \le 4$ and, by part 4, we obtain $4 = \mathsf{c}(H) \le \mathsf{c}_{\mathrm{mon}}(H)$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = 4$.

In order to finish the proof, we need an additional Lemma.

Lemma 3.18. *Let D be a monoid, $P \subset D$ be a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D_i}^{\times}$ be reduced half-factorial monoids of type $(1,1)$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, let $G = \mathsf{q}(D/H)$ be its class group with $|G| = 2$, say $G = \{\mathbf{0}, g\}$, suppose each class in G contains some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$. Furthermore, let $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ be the $(D_1 \times \ldots \times D_r)$-block monoid defined by ι over G and suppose $\mathcal{B}$ is half-factorial but not factorial.*
Then $\mathsf{t}(H) = \mathsf{t}(\mathcal{B}) = 2$.

Proof. Throughout the proof, we write $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_{r+s}, \iota) \mid \mathbf{0} \nmid S\}$ as in Lemma 3.4.4. By Proposition 3.17.1-4, we find that $\{i \in [1,r] \mid (\mathcal{U}_1(D_i))_{\mathbf{0}} \cap (\mathcal{U}_1(D_i))_g \neq \emptyset\} = \emptyset$, and thus $\iota(\widehat{D_i}^{\times}) = \{\mathbf{0}\}$ for all $i \in [1,r]$. Now let $h \in H$, $z \in \mathsf{Z}(h)$, and $a \in \mathcal{A}(H)$ be such that $a \mid h$. Then we prove that $\mathsf{d}(z,z') \le 2$ for some $z' \in \mathsf{Z}(h) \cap a\mathsf{Z}(H)$. We may assume that $a \nmid z$. We find that z is of the following form:

$$z = q_1 \cdot \ldots \cdot q_k (q_1^{(1)} q_1^{(2)}) \cdot \ldots \cdot (q_l^{(1)} q_l^{(2)}) t_1 \cdot \ldots \cdot t_m,$$

where $q_1, \ldots, q_k$, $q_1^{(1)}$, $q_1^{(2)}, \ldots, q_l^{(1)}$, $q_l^{(2)} \in P$, $[q_1]_{D/H} = \ldots = [q_k]_{D/H} = \mathbf{0}$, $[q_1^{(1)}]_{D/H} = [q_1^{(2)}]_{D/H} = \ldots = [q_l^{(1)}]_{D/H} = [q_l^{(2)}]_{D/H} = g$, and $t_1, \ldots, t_m \in \mathcal{A}(D_1 \times \ldots \times D_r)$. Now we have the following three possibilities for a:

$$\begin{aligned}
&a = \bar{q} && \text{with } \bar{q} \in P \text{ and } [\bar{q}]_{D/H} = \mathbf{0}, \text{ or} \\
&a = \bar{q}^{(1)} \bar{q}^{(2)} && \text{with } \bar{q}^{(1)} \text{ and } \bar{q}^{(2)} \in P,\ [\bar{q}^{(1)}]_{D/H} = [\bar{q}^{(2)}]_{D/H} = g, \text{ or} \\
&a = u && \text{with } u \in \mathcal{A}(D_i) \text{ for some } i \in [1,r].
\end{aligned}$$

We proceed case by case. Let $a = \bar{q}$, where $\bar{q} \in P$ and $[\bar{q}]_{D/H} = \mathbf{0}$. Since $q_1, \ldots, q_k$, $q_1^{(1)}$, $q_1^{(2)}, \ldots, q_l^{(1)}$, $q_l^{(2)} \in P$ are prime in D and since $[\bar{q}]_{D/H} = \mathbf{0}$, we find $\bar{q} \in \{q_1, \ldots, q_k\}$. Thus $a = \bar{q} \mid z$, a contradiction. Let $a = \bar{q}^{(1)} \bar{q}^{(2)}$, where $\bar{q}^{(1)}$, $\bar{q}^{(2)} \in P$ and $[\bar{q}^{(1)}]_{D/H} = [\bar{q}^{(2)}]_{D/H} = g$. By the same arguments as before, we find $\bar{q}^{(1)}$, $\bar{q}^{(2)} \in \{q_1^{(1)}, q_1^{(2)}, \ldots, q_l^{(1)}, q_l^{(2)}\}$. Since $a \nmid z$, there is no $i \in [1,l]$ such that without loss of generality $\bar{q}^{(j)} = \bar{q}_i^{(j)}$ for $j = 1,2$. Thus there are i, $j \in [1,l]$ with $i \neq j$ such that without loss of generality $\bar{q}^{(1)} = q_i^{(1)}$ and $\bar{q}^{(2)} = q_j^{(2)}$. Now we find the factorization $z' \in \mathsf{Z}(h)$,

$$z' = q_1 \cdot \ldots \cdot q_k (q_i^{(1)} q_j^{(2)})(q_j^{(1)} q_i^{(2)}) \prod_{s=1 s\neq i,j}^{s=l} (q_s^{(1)} q_s^{(2)}) t_1 \cdot \ldots \cdot t_m,$$

such that $\mathsf{d}(z,z') = 2$ and $a \mid z'$. Lastly, we consider the case $a = u$ with $u \in \mathcal{A}(D_i)$ for some $i \in [1,r]$. Then there are $u_1, \ldots, u_{\bar{m}} \in \mathcal{A}(D_1 \times \ldots \times D_r)$ such that

$$t_1 \cdot \ldots \cdot t_m = u u_1 \cdot \ldots \cdot u_{\bar{m}} \quad \text{and } \mathsf{d}_{D_1 \times \ldots \times D_r}(t_1 \cdot \ldots \cdot t_m, u u_1 \cdot \ldots \cdot u_{\bar{m}}) \le 2.$$

Now we find a factorization $z' \in \mathsf{Z}(h)$ by setting

$$z' = q_1 \cdot \ldots \cdot q_k (q_1^{(1)} q_1^{(2)}) \cdot \ldots \cdot (q_l^{(1)} q_l^{(2)}) u u_1 \cdot \ldots \cdot u_{\bar{m}},$$

and $\mathsf{d}(z,z') \le 2$ follows. □

6. Let $s = 0$ and $\iota(p_i) = \mathbf{0}$ for all $i \in [1,r]$. If H is not half-factorial, then $\mathsf{c}(H) \ge 3$ and therefore $\mathsf{t}(H) \ge 3$. Otherwise, if H is half-factorial, then $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}) = 2$, and therefore $I = \emptyset$ by points 1-4. Thus $\iota(u) = \iota(p_i) = \mathbf{0}$ for all $u \in \mathcal{A}(D_i)$ and $i \in [1,r]$, and any $a \in \mathcal{A}(\mathcal{B})$ is either of the form $a = g^2$ or $a = u$ with $u \in \mathcal{A}(D_i)$ for some $i \in [1,r]$. Since, by Lemma 3.8.2, $\mathsf{t}(D_i) = 2$ for all $i \in [1,r]$, we have $\mathsf{t}(\mathcal{B}) = 2$. Now the assertion follows by Lemma 3.18. □

The following example shows that the very special structure of D in the hypothesis of Lemma 3.18—in terms of Example 3.19, the structure T—is definitely necessary for the assertion of Lemma 3.18 to hold.

Example 3.19. Let P be a set of prime elements and let T be an atomic monoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated submonoid with class group $D/H = C_2$ such that each class in C_2 contains some $p \in P$. Let $\iota : T \to C_2$, $t \mapsto [t]_{D/H}$ be a homomorphism and $\mathcal{B}(C_2, T, \iota)$ the T-block monoid over C_2 defined by ι. Furthermore let $\mathsf{t}(\mathcal{B}(C_2, T, \iota)) = 2$.
This situation does not imply $\mathsf{t}(H) = 2$.

Proof. We write $C_2 = \{\mathbf{0}, g\}$ and we set $\mathcal{B} = \mathcal{B}(C_2, T, \iota)$ and denote by $\beta : H \to \mathcal{B}$ the block homomorphism of H and by $\bar{\beta} : \mathsf{Z}(H) \to \mathsf{Z}(\mathcal{B})$ the canonical extension of the block homomorphism.
By definition, it is sufficient to prove $\mathsf{t}(a, v) \geq 3$ for some $a \in H$ and some $v \in \mathcal{A}(H)$. Let $a \in H$ and $v \in \mathcal{A}(H)$.
We have the following four types of atoms of H which are not prime:

$$\begin{aligned} v &= p_1 p_2 && \text{with } p_1,\, p_2 \in P \text{ and } [p_1]_{D/H} = [p_2]_{D/H} = g \\ v &= pt && \text{with } p \in P,\, t \in T \text{ and } [p]_{D/H} = [t]_{D/H} = g \\ v &= t_1 t_2 && \text{with } t_1,\, t_2 \in \mathcal{A}(T) \text{ and } [t_1]_{D/H} = [t_2]_{D/H} = g \\ v &= t && \text{with } t \in \mathcal{A}(T) \text{ and } [t]_{D/H} = g \end{aligned}$$

Let $z \in \mathsf{Z}_H(a)$. Without loss of generality, we may assume that no prime element divides a. Then z is of the following form:

$$z = (p_1 p_2) \cdot \ldots \cdot (p_{l-1} p_l)(p_{l+1} s_1) \cdot \ldots \cdot (p_{l+m} s_m)(t_1 t_2) \cdot \ldots \cdot (t_{n-1} t_n) u_1 \cdot \ldots \cdot u_o.$$

Let $v = q_1 q_2$ be of the first type. Since all $p \in P$ are prime in D, we find $i,\, j \in [1, l+m]$ such that $p_i = q_1$ and $p_j = q_2$. Assume $i = l+1$ and $j = l+2$. Then we find

$$z' = (p_{l+1} p_{l+2})(p_1 s_1)(p_2 s_2)(p_1 p_2)^{-1}(p_{l+1} s_1)^{-1}(p_{l+2} s_2)^{-1} z.$$

Thus $\mathsf{d}(z, z') = 3$. If we apply $\bar{\beta}$ to z', we find

$$\bar{\beta}(z') = g^2 (g s_1)(g s_2)(g^2)^{-1}(g s_1)^{-1}(g s_2)^{-1} \bar{\beta}(z) = \bar{\beta}(z),$$

and thus $\mathsf{d}(\bar{\beta}(z), \bar{\beta}(z')) = 0$. □

Corollary 3.20. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ be reduced half-factorial monoids of type $(1,1)$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ its class group, and suppose each class in G contains some $p \in P$.*
Then the following are equivalent:

- $\mathsf{c}_{\mathrm{mon}}(H) \leq 2$.
- $\mathsf{c}(H) \leq 2$.
- *H is half-factorial.*

If, additionally, $[p_i]_{D/H} = \mathbf{0}_{D/H}$ for all $i \in [1, r]$—in particular, this is true if $|G| = 1$—then the following is also equivalent:

- $\mathsf{t}(H) \leq 2$.

Proof. By Lemma 3.14.3, $|G| \geq 3$ implies $\mathsf{c}(H) \geq 3$ and thus that H is never half-factorial. Thus we have $|G| \leq 2$. If $|G| = 2$, then the assertion follows by Proposition 3.17. If $|G| = 1$, the assertion follows by Lemma 3.14.2 and Lemma 3.14.1. □

Lemma 3.21. *Let $\mathcal{O}$ be a locally half-factorial order in an algebraic number field.*
Then there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$, $\mathcal{I}^*(\mathcal{O}) \cong D$, $\mathcal{O}^\bullet_{\mathrm{red}} \subset D$ is a saturated submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathsf{q}(D/\mathcal{O}^\bullet_{\mathrm{red}})$ is its class group, and each class contains some $p \in P$.
If, additionally, all localizations of $\mathcal{O}$ are finitely primary monoids of exponent 1, then $k_i = 1$ for all $i \in [1, r]$.

Proof. Let $\mathcal{O}$ be an order in an algebraic number field and suppose $\mathcal{I}^*(\mathcal{O})$ is half-factorial. We set $\overline{\mathcal{O}}$ for the integral closure of $\mathcal{O}$ and set $\mathfrak{f} = (\mathcal{O} : \overline{\mathcal{O}})$, $\mathcal{P} = \{p \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \not\supset \mathfrak{f}\}$, $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset \mathfrak{f}\}$, and

$$T = \prod_{\mathfrak{p} \in \mathcal{P}^*} (\mathcal{O}^\bullet_{\mathfrak{p}})_{\mathrm{red}}.$$

By [12, Theorem 3.7.1], we find that $\mathcal{P}^*$ is finite, $\mathcal{O}^\bullet_{red} \subset \mathcal{F}(\mathcal{P}) \times T$ is a saturated and cofinal submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathcal{C}_{\mathsf{v}}(\mathcal{O}) = (\mathcal{F}(\mathcal{P}) \times T)/\mathcal{O}^\bullet_{red}$, and, for all $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$, $\mathcal{O}^\bullet_{\mathfrak{p}}$ is a finitely primary monoid of rank $s_{\mathfrak{p}}$, where $s_{\mathfrak{p}}$ is the number of prime ideals $\overline{\mathfrak{p}} \in \mathfrak{X}(\overline{\mathcal{O}})$ such that $\overline{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}$. For $\mathfrak{p} \in \mathcal{P}^*$, the local domain $\mathcal{O}_{\mathfrak{p}}$ is not integrally closed, hence not factorial, and therefore the monoid $(\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ is not factorial either. Since $\mathcal{I}^*(\mathcal{O}) \cong \prod_{\mathfrak{p}\in\mathfrak{X}(\mathcal{O})}(\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ by [12, Theorem 3.7.1], we find, for all $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$, that $\mathcal{O}_\mathfrak{p}$ is half-factorial, and thus, by the additional statement in Lemma 3.8.1, $\mathcal{O}_\mathfrak{p}^\bullet$ is a half-factorial monoid of type $(1, k_\mathfrak{p})$, where $k_\mathfrak{p}$ is the rank of $\mathcal{O}_\mathfrak{p}^\bullet$. By [18, Corollary 3.5], we find $k_\mathfrak{p} \le 2$. Now we set $D_i = (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ for some $\mathfrak{p} \in \mathcal{P}^*$ such that $T = D_1 \times \ldots \times D_r$ and we set $P = \mathcal{P}$. By [12, Corollary 2.11.16], every class contains infinitely many primes $p \in P$. Since, by the above, k_i is the exponent of D_i for all $i \in [1, r]$, the additional statement is obvious. □

The final proof of the main theorem.

Final proof of Theorem 1.1. By Lemma 3.21, there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^\times = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$, $\mathcal{I}^*(\mathcal{O}) \cong D$, $\mathcal{O}_{\mathrm{red}}^\bullet \subset D$ is a saturated submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathsf{q}(D/\mathcal{O}_{\mathrm{red}}^\bullet)$ is its class group, and each class contains some $p \in P$.

1. If $|\mathrm{Pic}(\mathcal{O})| = 1$, then the assertion follows by Lemma 3.14.2.
2. If $|\mathrm{Pic}(\mathcal{O})| \ge 3$, then the assertion follows by Lemma 3.14.3.
3. If $|\mathrm{Pic}(\mathcal{O})| = 2$, then $\rho(\mathcal{O}) \le 2$ and $2 \le \mathsf{c}(\mathcal{O}) \le 4$ follow by Lemma 3.14.4. If, additionally, all localizations of $\mathcal{O}$ are finitely primary monoids of exponent 1, then, by Lemma 3.21, we have $k_i = 1$ for all $i \in [1, r]$. If $k = 0$, then we are in the situation of Proposition 3.17.1, and thus $\mathcal{O}$ is half-factorial, $\mathsf{c}(\mathcal{O}) = 2$, and $\triangle(\mathcal{O}) = \emptyset$. If $k \ge 2$, then we are in the situation of Proposition 3.17.4, and thus $\rho(\mathcal{O}) = 2$, $\mathsf{c}(\mathcal{O}) = 4$, and $\triangle(\mathcal{O}) = \{1, 2\}$. If $k = 1$, then we are in the situation of Proposition 3.17.3, and thus $\rho(\mathcal{O}) \ge \frac{3}{2}$, $\mathsf{c}(\mathcal{O}) = 3$, and $\triangle(\mathcal{O}) = \{1\}$. Since $k_i = 1$ for all $i \in [1, r]$, we may use Proposition 3.17.5. Thus we find $\rho(\mathcal{O}) = \frac{3}{2}$ if $k = 1$ and $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = \mathsf{c}(\mathcal{O})$ in all cases. Putting all this together, we obtain the formulas in the assertion. The equivalence of the four statements follows by Corollary 3.20.

In particular, in all situations, we find $\min \triangle(\mathcal{O}) \le 1$. □

4. Consequences and refinements of the main theorem

In the case of quadratic and cubic number fields, we can do even better. First, we recall and reformulate a definition and the key result from [18].

Let $\mathcal{O}$ be an order in an algebraic number field and $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$. Then we call $\mathcal{O}_\mathfrak{p}$ a local order. Now let $\mathcal{O}_\mathfrak{p}$ be a local order such that its integral closure $\overline{\mathcal{O}_\mathfrak{p}}$ is local too. Now we fix the following notations. We denote by $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ the maximal ideal of $\mathcal{O}_\mathfrak{p}$ respectively $\overline{\mathcal{O}_\mathfrak{p}}$, by $k = \mathcal{O}_\mathfrak{p}/\mathfrak{m}$ and $\overline{k} = \overline{\mathcal{O}_\mathfrak{p}}/\overline{\mathfrak{m}}$ the residue class fields, and by $\pi : \overline{\mathcal{O}_\mathfrak{p}} \to \overline{k}$ the canonical homomorphism. For a prime $p \in \overline{\mathcal{O}_\mathfrak{p}}$ and $i \in \mathbb{N}$, we set

$$U_{i,p}(\mathcal{O}_\mathfrak{p}) = \{\varepsilon \in \overline{\mathcal{O}_\mathfrak{p}}^\times \mid \varepsilon p^i \in \mathcal{O}_\mathfrak{p}\} \quad \text{and} \quad V_{i,p}(\mathcal{O}_\mathfrak{p}) = \pi(U_{i,p}(\mathcal{O}_\mathfrak{p})) \cup \{0\},$$

as in [18]. Then $V_{i,p}(\mathcal{O}_\mathfrak{p})$ is a k-subspace of $\overline{k}$ by [18].

Lemma 4.1 ([18, Theorem 3.3])**.** *Using the above notations, the following are equivalent:*

1. *$\mathcal{O}_\mathfrak{p}$ is half-factorial.*
2. *$(U_{1,p}(\mathcal{O}_\mathfrak{p}))^2 = \overline{\mathcal{O}_\mathfrak{p}}^\times$.*
3. *$\{xy \mid (x, y) \in V_{1,p}(\mathcal{O}_\mathfrak{p}) \times V_{1,p}(\mathcal{O}_\mathfrak{p})\} = \overline{k}$.*

Lemma 4.2. *Let $\mathcal{O}$ be an order in an algebraic number field and $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ such that $\mathcal{O}_\mathfrak{p}$ is half-factorial.*

1. *$\overline{\mathcal{O}_\mathfrak{p}}$ is local and every atom of $\mathcal{O}_\mathfrak{p}$ is a prime of $\overline{\mathcal{O}_\mathfrak{p}}$.*
2. *Let $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ be the maximal ideals of $\mathcal{O}_\mathfrak{p}$ respectively $\overline{\mathcal{O}_\mathfrak{p}}$ and let $k = \mathcal{O}_\mathfrak{p}/\mathfrak{m}$ and $\overline{k} = \overline{\mathcal{O}_\mathfrak{p}}/\overline{\mathfrak{m}}$ be the residue class fields.*
   If $\dim_k \overline{k} \le 3$, then $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}}^\bullet$ is a finitely primary monoid of exponent 1.
   In particular, if $\mathcal{O}$ is an order in a quadratic or cubic number field, then $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}}^\bullet$ is a finitely primary monoid of exponent 1.

Whenever $\mathcal{O}_\mathfrak{p}$ is a Cohen-Kaplansky domain, i.e., whenever it has up to units only finitely many atoms, the result from Lemma 4.2.1 can be found in [1, Theorem 6.3].

Proof of Lemma 4.2.

1. The assertion follows by Lemma 3.8.1.

2. By part 1, $\overline{\mathcal{O}_\mathfrak{p}}$ is local too. Thus $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ is well-defined and k respectively $\overline{k}$ is a field. Since $\overline{\mathcal{O}_\mathfrak{p}}$ has up to units only one prime element by part 1, we write $V_1(\mathcal{O}_\mathfrak{p})$ instead of $V_{1,p}(\mathcal{O}_\mathfrak{p})$ and $U_1(\mathcal{O}_\mathfrak{p})$ instead of $U_{1,p}(\mathcal{O}_\mathfrak{p})$. Furthermore, we find $\mathcal{U}_1(\mathcal{O}_\mathfrak{p}^\bullet) = U_1(\mathcal{O}_\mathfrak{p})$. For short, we write $m = \dim_k \overline{k}$, $n = \dim_k V_1(\mathcal{O}_\mathfrak{p})$, and $q = \#k$. Now we distinguish three cases by m.
   Case 1 $m = 1$. Here $k = \overline{k}$ and therefore $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$. Thus $U_1(\mathcal{O}_\mathfrak{p}) = \overline{\mathcal{O}_\mathfrak{p}}^\times$ by [18, Lemma 3.2], and therefore $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}^\bullet}$ is of exponent 1 by the additional statement of Lemma 3.8.1.
   Case 2 $m = 2$. If $n = 1$, then $V_1(\mathcal{O}_\mathfrak{p}) = k$, and therefore $V_1(\mathcal{O}_\mathfrak{p}) * V_1(\mathcal{O}_\mathfrak{p}) = k \neq \overline{k}$, a contradiction to Lemma 4.1.3. If $n = 2$, then $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$, and the assertion follows as in Case 1.
   Case 3 $m = 3$. If $n = 1$, then we find the same contradiction as in Case 2 when $n = 1$ there. If $n = 2$, then $\#(V_1(\mathcal{O}_\mathfrak{p}) * V_1(\mathcal{O}_\mathfrak{p})) < q^3 = \#\overline{k}$ by [18, Lemma 2.5]. This is again a contradiction to Lemma 4.1.3. If $n = 3$, then $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$, and the assertion follows as in Case 1.
   Let K be the algebraic number field containing $\mathcal{O}$. Then we find $m \le [K : \mathbb{Q}]$ and the assertion follows. □

Now we can prove a slightly refined version of Theorem 1.1 for orders in quadratic and cubic number fields.

Corollary 4.3. *Let $\mathcal{O}$ be a non-principal, locally half-factorial order in a quadratic or cubic number field and set $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \overline{\mathcal{O}})\}$.*

1. *If $|\operatorname{Pic}(\mathcal{O})| = 1$, then $\mathcal{O}$ is half-factorial.*
2. *If $|\operatorname{Pic}(\mathcal{O})| \ge 3$, then $(\mathsf{D}(\operatorname{Pic}(\mathcal{O})))^2 \ge \mathsf{c}(\mathcal{O}) \ge 3$, and $\min \triangle(\mathcal{O}) = 1$.*
3. *If $|\operatorname{Pic}(\mathcal{O})| = 2$, then, setting $k = \#\{\mathfrak{p} \in \mathcal{P}^* \mid [\overline{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = \operatorname{Pic}(\mathcal{O})\}$, it follows that*
   - $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(\mathcal{O}) = 2 + \min\{2, k\} \in \{2, 3, 4\}$;
   - $\rho(\mathcal{O}) = \frac{1}{2}\mathsf{c}(\mathcal{O}) \in \{1, \frac{3}{2}, 2\}$;
   - $\triangle(\mathcal{O}) = [1, \mathsf{c}(\mathcal{O}) - 2] \subset [1, 2]$.

In particular, $\min \triangle(\mathcal{O}) \le 1$ always holds, and the following are equivalent:

- $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = 2$.
- $\mathsf{c}(\mathcal{O}) = 2$.
- *$\mathcal{O}$ is half-factorial.*

If, additionally, $[\mathfrak{p}] = \mathbf{0}_{\operatorname{Pic}(\mathcal{O})}$ for all $\mathfrak{p} \in \mathcal{P}^*$—this is always true if $|\operatorname{Pic}(\mathcal{O})| = 1$ or if $\mathcal{O}$ is an order in a quadratic number field—then the following is also equivalent:

- $\mathsf{t}(\mathcal{O}) = 2$.

Proof. Part 1 respectively part 2 follows immediately from Theorem 1.1.1 respectively Theorem 1.1.2. By Lemma 4.2.2, all localizations $\mathcal{O}_\mathfrak{p}$ for $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ are finitely primary monoids of exponent (at most) 1. Thus part 3 follows by the additional statement of Theorem 1.1.3.
Now we prove the additional statement. First note $\min \triangle(\mathcal{O}) \le 1$ follows by the additional statement of Theorem 1.1. If $|\operatorname{Pic}(\mathcal{O})| \ge 3$, then none of the equivalent conditions holds by part 2. If $|\operatorname{Pic}(\mathcal{O})| = 2$, then the four equivalent conditions are shown in the additional statement of Theorem 1.1.3. If $|\operatorname{Pic}(\mathcal{O})| = 1$, then $\mathcal{O} \cong \mathcal{I}^*(\mathcal{O})$, and therefore $\mathcal{O}$ is half-factorial. By Lemma 3.21 and Lemma 4.2.2, there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^\times = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$ and $\mathcal{I}^*(\mathcal{O}) \cong D$. Now the other equivalent conditions follow by Lemma 3.8.2. □

If we compare the equivalent conditions in Corollary 4.3 for non-principal, locally half-factorial orders in quadratic or cubic number fields with the ones given in [12, Theorem 1.7.3.6]—see below for principal orders in algebraic number fields—we see that at least these special non-principal orders behave nearly the same as the principal ones.

Theorem 4.4 (cf. [12, Theorem 1.7.3.6])**.** *Let $\mathcal{O}$ be a principal order in a quadratic or cubic number field. Then the following are equivalent.*

1. *$\mathcal{O}$ is half-factorial.*
2. *$|\operatorname{Pic}(\mathcal{O})| \le 2$.*
3. *$\mathsf{t}(\mathcal{O}) \le 2$.*
4. *$\mathsf{c}(\mathcal{O}) \le 2$.*

By Corollary 4.3.3, we get an additional bound on the elasticity of a non-principal order $\mathcal{O}$ in a quadratic or cubic number field such that its conductor is an inert prime ideal, say $(\mathcal{O} : \overline{\mathcal{O}}) = \mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ and $\mathfrak{p}\overline{\mathcal{O}} \in \mathfrak{X}(\overline{\mathcal{O}})$; then $\rho(\mathcal{O}) \le \frac{3}{2}$. Now we revisit the example from [15, example at the end of the publication]:

Let $\mathcal{O} = \mathbb{Z}[3i]$. Then $\overline{\mathcal{O}} = \mathbb{Z}[i]$, $|\operatorname{Pic}(\mathcal{O})| = 2$, $\mathcal{O}$ is locally half-factorial, and $(\mathcal{O} : \overline{\mathcal{O}}) = 3\mathcal{O} \in \mathfrak{X}(\mathcal{O})$ is an inert prime ideal in $\overline{\mathcal{O}}$. We set $\beta = 1 + 2i$ and $\beta' = 1 - 2i$. Then 3β, $3\beta'$, 3, and 5 are irreducible elements of $\mathcal{O}$ satisfying $(3\beta)(3\beta') = 3^2 \cdot 5$; thus $\rho(\mathcal{O}) \geq \frac{3}{2}$. Now we have equality by Corollary 4.3.3.

4.1. Localizations of half-factorial orders.

Proposition 4.5. *Let D be a monoid, $P \subset D$ be a set of prime elements, and let $T \subset D$ be a reduced atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $D_1 \subset T$ be a divisor-closed submonoid and let $D_1 \subset [p] \times \widehat{D_1}^{\times} = \widehat{D_1}$ be a finitely primary monoid of rank 1 and exponent k. Let $H \subset D$ be a saturated half-factorial submonoid, $G = \mathsf{q}(D/H)$ its class group, and suppose each class in G contains some $p' \in P$. Then $|G| \leq 2$ and D_1 is either*

- *half-factorial or*
- *$|G| = 2$, say $G = \{\mathbf{0}, g\}$, $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1, 2\}$, $[p]_{D/H} = g$, and $[\varepsilon]_{D/H} = \mathbf{0}$ for all $\varepsilon \in \widehat{D_1}^{\times}$.*

Proof. Define a homomorphism $\iota : T \to G$ by $\iota(t) = [t]_{D/H}$. Throughout the proof, we write $\mathcal{B} = \{S \in \mathcal{B}(G, T, \iota) \mid \mathbf{0} \nmid S\}$ as in Lemma 3.4.4. If $|G| \geq 3$, then it follows by Lemma 3.5.1 that H is not half-factorial. If $|G| = 1$, then $H = D$ and, obviously, the first case of the assertion holds. Now, let $|G| = 2$, say $G = \{\mathbf{0}, g\}$. Since H is half-factorial, $\mathcal{B}$ is also half-factorial by Lemma 3.4. By Lemma 3.11, $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$ is equivalent to D_1 being half-factorial. We show that either

- $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$ or
- $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1, 2\}$, $\iota(p) = g$, and $\iota(\widehat{D_1}^{\times}) = \{\mathbf{0}\}$.

If $\#\mathsf{v}_p(\mathcal{A}(D_1)) = 1$, i.e., $\mathsf{v}_p(\mathcal{A}(D_1)) = \{n\}$, then we find $n = 1$ since $N_{\geq k} \subset n\mathbb{N}_0$. Suppose we have $\#\mathsf{v}_p(\mathcal{A}(D_1)) > 1$. Then there are $n = \min \mathsf{v}_p(\mathcal{A}(D_1))$ and $m = \max \mathsf{v}_p(\mathcal{A}(D_1)) > n$. Let $\varepsilon, \eta \in \widehat{D_1}^{\times}$ be such that $p^n\varepsilon$, $p^m\eta \in \mathcal{A}(D_1)$. Now we distinguish four cases by $\iota(p^n\varepsilon)$ and $\iota(p^m\eta)$.

Case 1 $\iota(p^n\varepsilon) = \iota(p^m\eta) = \mathbf{0}$. Then $p^n\varepsilon$, $p^m\eta \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta)^k = (p^n\varepsilon)^k(p^{(m-n)k}\eta^k\varepsilon^{-k}).$$

There are k atoms on the left side and at least $k+1$ on the right side; clearly a contradiction to $\mathcal{B}$ being half-factorial.

Case 2 $\iota(p^n\varepsilon) = \iota(p^m\eta) = g$. Then $p^n\varepsilon g$, $p^m\eta g \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta g)^k = (p^n\varepsilon g)^k(p^{(m-n)k}\eta^k\varepsilon^{-k})$$

There are k atoms on the left side and at least $k+1$ on the right side; clearly a contradiction to $\mathcal{B}$ being half-factorial.

Case 3 $\iota(p^n\varepsilon) = \mathbf{0}$ and $\iota(p^m\eta) = g$. Then $p^n\varepsilon$, $p^m\eta g \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta g)^k = \begin{cases} (p^n\varepsilon)^k(g^2)^{\frac{k}{2}}(p^{(m-n)k}\eta^k\varepsilon^{-k}) & k \text{ even} \\ (p^n\varepsilon)^k(g^2)^{\frac{k-1}{2}}(p^{(m-n)k}\eta^k\varepsilon^{-k}g) & k \text{ odd.} \end{cases}$$

There are k atoms on the left side and, in both cases, at least $k + 1 + \frac{k-1}{2}$ on the right side; clearly a contradiction to $\mathcal{B}$ being half-factorial.

Case 4 $\iota(p^n\varepsilon) = g$ and $\iota(p^m\eta) = \mathbf{0}$. Then $p^n\varepsilon g$, $p^m\eta \in \mathcal{A}(\mathcal{B})$. Now we must again distinguish four cases.

Case 4.1 $2n < m$ and k even. Here we find

$$(g^2)^{\frac{k}{2}}(p^m\eta)^k = (p^n\varepsilon g)^k(p^{(m-n)k}\varepsilon^{-k}\eta^k).$$

There are $\frac{5}{2}k$ atoms on the left side and at least $k + \left\lceil \frac{(m-n)k}{m} \right\rceil$ atoms on the right side. This is a contradiction to $\mathcal{B}$ being half-factorial, since $m > 2n$ by assumption.

Case 4.2 $2n < m$ and k odd. Here we find

$$(g^2)^{\frac{k+1}{2}}(p^m\eta)^{k+1} = (p^n\varepsilon g)^{k+1}(p^{(m-n)(k+1)}\varepsilon^{-k-1}\eta^{k+1}).$$

This leads to a contradiction as in the case where k was even.

Case 4.3 $m < 2n$ and k even. We choose $l \in \mathbb{N}$ maximal with $lm \leq (n-1)k$, and we find

$$(p^n\varepsilon g)^k = (g^2)^{\frac{k}{2}}(p^{nk-lm}\varepsilon^k\eta^{-l})(p^m\eta)^l.$$

There are k atoms on the left side and at least $\frac{k}{2} + \left\lceil \frac{nk-lm}{m} \right\rceil + l$ on the right. This is a contradiction to $\mathcal{B}$ being half-factorial, since $m < 2n$ by assumption.

Case 4.4 $m < 2n$ and k odd. We choose $l \in \mathbb{N}$ maximal with $lm \leq (n-1)(k+1)$, and we find a contradiction to $\mathcal{B}$ being half-factorial by looking at

$$(p^n\varepsilon g)^{k+1} = (g^2)^{\frac{k+1}{2}}(p^{n(k+1)-lm}\varepsilon^{k+1}\eta^{-l})(p^m\eta)^l.$$

Case 4.5 $m = 2n$. In this particular case, we must again handle two additional cases.

Case 4.5.1 $n > 1$. Then there is $n' \in (n, 2n)$ and $\gamma \in \widehat{D}_1^\times$ such that $p^{n'}\gamma \in \mathcal{A}(D_1)$. If $\iota(p^{n'}\gamma) = \mathbf{0}$, then the assertion follows with $p^{n'}\gamma$ and $p^m\eta$ as in Case 1. If $\iota(p^{n'}\gamma) = g$, then the assertion follows with $p^n\varepsilon$ and $p^{n'}\gamma$ as in Case 2.

Case 4.5.2 $n = 1$. Then $m = 2n = 2$. Without loss of generality we may assume that $p \in D_1$. Furthermore, $\iota(p^2\eta) = \mathbf{0}$ implies $\iota(\eta) = \mathbf{0}$. For the moment, we assume that $\iota(p) = \mathbf{0}$ and $\iota(\varepsilon) = g$. Then we are done by Case 1 with p and $p^2\eta$. If now $\iota(p) = g$ and $\iota(\varepsilon) = \mathbf{0}$, we show $\iota(\widehat{D_1}^\times) = \{\mathbf{0}\}$ or $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$. If $\iota(\widehat{D_1}^\times) = \{\mathbf{0}\}$, then the second case in the assertion is fulfilled. Now suppose $\iota(\widehat{D_1}^\times) = G$, say there is some $\gamma \in \widehat{D_1}^\times$ with $\iota(\gamma) = g$. Then there is some $k' \in [1, k]$ such that $p^{k'}\gamma \in D_1$. Thus there are $\varepsilon_1, \ldots, \varepsilon_l, \eta_1, \ldots, \eta_{l'} \in \widehat{D_1}^\times$ such that $(p\varepsilon_1) \cdot \ldots \cdot (p\varepsilon_l)(p^2\eta_1) \cdot \ldots \cdot (p^2\eta_{l'}) = p^{k'}\gamma$ is a factorization of $p^{k'}\gamma$ in D_1. Thus $\varepsilon_1 \cdot \ldots \cdot \varepsilon_l\eta_1 \cdot \ldots \cdot \eta_{l'} = \gamma$, and therefore either $\iota(\varepsilon_i) = g$ for some $i \in [1, l]$ or $\iota(\eta_j) = g$ for some $j \in [1, l']$. In the first case, we are in the situation of Case 1 with $p\varepsilon_i$ and $p^2\eta$, and in the second case, we are in the situation of Case 2 with p and $p^2\eta_j$. □

Corollary 4.6. *Let $\mathcal{O}$ be a half-factorial order in an algebraic number field K, $\mathcal{O}_K$ is integral closure, and let $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ be a prime ideal of $\mathcal{O}$ such that $\mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)$.*
Then $|\operatorname{Pic}(\mathcal{O})| \le 2$ and $\mathcal{O}_\mathfrak{p}$ is either

- *half-factorial, and $\mathcal{O}_\mathfrak{p}^\bullet \subset (\mathcal{O}_K)_\mathfrak{p}^\bullet$ is a half-factorial monoid of type $(1, k)$ with $k \in \{1, 2\}$, or*
- *$\mathfrak{p}$ ramifies in $\mathcal{O}_K$ with ramification degree 2, i.e. there is some $\overline{\mathfrak{p}} \in (\mathcal{O}_K)_\mathfrak{p}$ prime such that $\overline{\mathfrak{p}}^2 \sim \mathfrak{p}$.*

In particular, if K is a quadratic number field, then $\mathcal{O}_\mathfrak{p}$ is half-factorial.

Proof. Let $\mathcal{O}$ be a half-factorial order in an algebraic number field K, let $\mathcal{O}_K$ be its integral closure, let $\mathcal{P} = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \not\supset (\mathcal{O} : \mathcal{O}_K)\}$, and let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)\}$. By [12, Theorem 3.7.1], we find that

$$\mathcal{O}_{\mathrm{red}}^\bullet \subset \mathcal{F}(\mathcal{P}) \times T \quad \text{with } T = \prod_{\mathfrak{p} \in \mathcal{P}^*} (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$$

is a saturated cofinal submonoid with class group $\operatorname{Pic}(\mathcal{O})$. Now, we obtain $|\operatorname{Pic}(\mathcal{O})| \le 2$ by Lemma 3.5.1. Since $\mathcal{O}$ is half-factorial, i.e., $\rho(\mathcal{O}) = 1 < \infty$, we find, by [15, Corollary 4.i], that $\mathfrak{p}$ does not split in $\mathcal{O}_K$. Thus $(\mathcal{O}_K)_\mathfrak{p}$ is a discrete valuation domain, in particular, it is local, and thus $\mathcal{O}_\mathfrak{p}^\bullet \subset (\mathcal{O}_K)_\mathfrak{p}^\bullet$ is a finitely primary monoid of rank 1. Since $(\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}} \subset T$ is a divisor-closed submonoid, the assertion follows immediately by Proposition 4.5.
If K is a quadratic number field, then $\mathcal{O}$ being half-factorial implies that $\mathfrak{p}$ is inert by [12, First paragraph from the Proof of A2 in the Proof of Theorem 3.7.15], and therefore $\mathcal{O}_\mathfrak{p}$ is half-factorial. □

4.2. Characterization of half-factorial orders in quadratic number fields.

Corollary 4.7. *Let $\mathcal{O}$ be a non-principal order in a quadratic number field K, let $\mathcal{O}_K$ be its integral closure, and let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)\}$.*
Then the following are equivalent:

1. *$\mathcal{O}$ is half-factorial.*
2. *$\mathsf{c}(\mathcal{O}) = 2$.*
3. *$|\operatorname{Pic}(\mathcal{O})| \le 2$, $\mathcal{O}$ is locally half-factorial and, for all $\mathfrak{p} \in \mathcal{P}^*$, $[(\mathcal{O}_K)_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = [\mathbf{0}]_{\operatorname{Pic}(\mathcal{O})}$.*
4. *$|\operatorname{Pic}(\mathcal{O})| \le 2$ and, for all $\mathfrak{p} \in \mathcal{P}^*$,*
   - *$[(\mathcal{O}_K)_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = [\mathbf{0}]_{\operatorname{Pic}(\mathcal{O})}$,*
   - *$\mathfrak{p}$ is inert in $\mathcal{O}_K$, and*
   - *$\mathfrak{p}^2 \not\supset (\mathcal{O} : \mathcal{O}_K)$.*

Proof. **1** ⇔ **2** By Corollary 4.6, we reason $\mathcal{I}^*(\mathcal{O})$ is half-factorial. Thus the assertion is already shown in the additional statement of Corollary 4.3.
1 ⇒ **3** By Corollary 4.6, $|\operatorname{Pic}(\mathcal{O})| \le 2$ and $\mathcal{O}_\mathfrak{p}$ is half-factorial for all $\mathfrak{p} \in \mathcal{P}^*$. We get $[(\mathcal{O}_K)_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = [\mathbf{0}]_{\operatorname{Pic}(\mathcal{O})}$ by the same construction as in the proof of Corollary 4.3 and Theorem 1.1 using Proposition 3.17.
3 ⇒ **1** Since, by assumption, $\mathcal{O}$ is locally half-factorial, this implication follows, directly, by the same construction as in the proof of Corollary 4.3 and Theorem 1.1 using Proposition 3.17.
3 ⇔ **4** Since, for all $\mathfrak{p} \in \mathcal{P}^*$, $\mathcal{O}_\mathfrak{p}$ is half-factorial if and only if $\mathfrak{p}$ is inert in $\mathcal{O}_K$ and $\mathfrak{p}^2 \not\supset (\mathcal{O} : \mathcal{O}_K)$, the assertion follows. □

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