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Non-unique factorizations
—
A semigroup-theoretic algorithmic approach with applications to non-principal orders in algebraic number fields

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CHAPTER 1

Introduction and preliminaries

1.1. Introduction

The maximal order $\mathcal{O}_K$ of an algebraic number field is a Dedekind domain, and its arithmetic is completely determined by its Picard group $\mathrm{Pic}(\mathcal{O}_K)$. In particular, $\mathcal{O}_K$ is factorial if and only if its Picard group is trivial and $\mathcal{O}_K$ is half-factorial if and only if $\#\,\mathrm{Pic}(\mathcal{O}_K) \leq 2$; for reference see [**4**]. In contrast, non-principal orders are not integrally closed, hence they are never factorial, and their arithmetic depends not only on their Picard group but also on the localizations at singular primes. Though, a non-principal order $\mathcal{O}$ with $\#\,\mathrm{Pic}(\mathcal{O}) \geq 3$ inherits many arithmetical properties from the maximal order; in particular it cannot be half-factorial, see Theorem 3.1.18. In contrast, only little is known about the arithmetic of non-principal orders whose Picard group has at most two elements, even if they are locally half-factorial—see Definition 1.2.12 for a precise definition. Exactly in this situation we formulate our main results in Chapter 3—see Theorem 3.1.18, Corollary 3.1.21, and, for quadratic number fields, Corollary 3.1.25—by using the semi-group theoretic approach from Chapter 2.

In Chapter 2, the monoid of relations is used to study various invariants of non-unique factorizations. These are the elasticity, the tame degree, the catenary degree, and the monotone catenary degree; for a statement of the formal definitions, see Section 1.2; and for recent work on the catenary degree, see, for example, [**13**], and for former work on the monotone catenary degree, see [**10**], [**11**], and [**12**]. The monoid of relations associated to a monoid and a certain invariant $\mu(\cdot)$ have been used to study all these invariants but the monotone catenary degree. Investigations of this type started only fairly recently. In [**5**], such investigations were carried out for finitely generated monoids using the results from [**7**] and [**26**]. In [**6**] and [**19**], these results, and expansions thereof, were applied in the investigation of numerical monoids, which are (certain) finitely generated submonoids of the non-negative integers; for a detailed exposition of the theory of numerical monoids and applications, see, e.g., the monograph [**24**]. Even more recently, in [**22**]—included as Section 2.1 with slight changes in the notation in this thesis—additionally, these methods were extended to general not necessarily finitely generated monoids. In [**3**], the monotone catenary degree was studied. Now Section 2.2 extends this approach by applying the same methods as in [**22**] and in Section 2.1 in the computation of the monotone catenary degree. Based on these results, we can formulate an algorithmic approach for the computation of the studied invariants in the situation of T-block monoids in Section 2.3.

In Chapter 3, we start by investigating half-factorial finitely primary monoids. These turn out to have a very nice structure; see Lemma 3.1.3. Then we exploit this structure together with results from Chapter 2 to describe precisely the arithmetic of (non-principal) locally half-factorial orders in algebraic number fields—see Definition 1.2.12 for the precise definition; see Theorem 3.1.18. Even more detailed results can be formulated for the case of quadratic or cubic number fields; see Corollary 3.1.21. Then localizations of half-factorial orders in algebraic number fields are studied and a characterization of half-factorial orders in quadratic number fields in terms of the class group and the image of the unit groups of the localizations in the class group is given; see Corollary 3.1.25. In Section 3.2, first results for locally half-factorial orders with cyclic class groups are formulated; see Corollary 3.2.5.

In Chapter 4, we use some special type of finitely primary monoids, so called strict monoids—for a statement of the formal definition, see Definition 4.1.3—to calculate the minimum distance in special orders in algebraic function fields and in algebraic number fields.

1.2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \uplus \{0\}$. For integers n, $m \in \mathbb{Z}$, we set $[n, m] = \{x \in \mathbb{Z} \mid n \le x \le m\}$. By convention, the supremum of the empty set is zero and we set $\frac{0}{0} = 1$. The term "monoid" always means a commutative, cancellative semigroup with unit element. For a monoid H, we denote by $H^\times$ the set of invertible elements of H. If, for two elements a, $b \in H$, there is some $u \in H^\times$ such that $a = ub$, then we call a and b *associated*. In this case, we write $a \sim b$. We call H reduced if $H^\times = \{1\}$ and call $H_{\mathrm{red}} = H/H^\times$ the reduced monoid associated with H. Of course, H_{red} is always reduced, and the arithmetic of H is determined by H_{red}. Let H be an atomic monoid. We denote by $\mathcal{A}(H)$ its set of atoms, by $\mathcal{A}(H_{\mathrm{red}})$ the set of atoms of H_{red}, by $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ the free monoid with basis $\mathcal{A}(H_{\mathrm{red}})$, and by $\pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the unique homomorphism such that $\pi_H | \mathcal{A}(H_{\mathrm{red}}) = \mathrm{id}$. We call $\mathsf{Z}(H)$ the *factorization monoid* and π_H the *factorization homomorphism* of H. For $a \in H$, we denote by $\mathsf{Z}(a) = \pi_H^{-1}(aH^\times)$ the *set of factorizations* of a and denote by $\mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\}$ the *set of lengths* of a. We call $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ the *system of sets of lengths* of H. A monoid H is called *half-factorial*, if $\#\mathsf{L}(a) = 1$ for all $a \in H$, and *factorial* if $\#\mathsf{Z}(a) = 1$ for all $a \in H$.

In the following, we briefly recall the definitions of all the invariants of non-unique factorization to be dealt with in this thesis.

Definition 1.2.1. A monoid H is called a *BF-monoid* (or equivalently a *monoid with bounded factorizations*) if H is atomic and $\mathsf{L}(a)$ is finite for every $a \in H$.

Definition 1.2.2. Let H be an atomic monoid. For $a \in H$, we set

$$\rho(a) = \frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} \quad \text{and call} \quad \rho(H) = \sup\{\rho(a) \mid a \in H\} \quad \text{the } \textit{elasticity} \text{ of } H.$$

Note that H is half-factorial if and only if $\rho(H) = 1$.

Definition 1.2.3. Let H be an atomic monoid and z, $z' \in \mathsf{Z}(H)$ be two factorizations. Then we call

$$\mathsf{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z, z')}\right|, \left|\frac{z'}{\gcd(z, z')}\right|\right\} \quad \text{the } \textit{distance} \text{ between } z \text{ and } z'.$$

Definition 1.2.4. Let H be an atomic monoid.

1. For $a \in H$, the *catenary degree* $\mathsf{c}(a)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any two factorizations z, $z' \in \mathsf{Z}(a)$, there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, and $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$.

If this is the case, we say that z and z' can be concatenated by an N-chain. Also, $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\}$ is called the *catenary degree* of H.

2. For $a \in H$, the *monotone catenary degree* $\mathsf{c}_{\mathrm{mon}}(a)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

   For any two factorizations z, $z' \in \mathsf{Z}(a)$ with $|z| \le |z'|$, there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, $\mathsf{d}(z_{i-1}, z_i) \le N$ and $|z_{i-1}| \le |z_i|$ for all $i \in [1, k]$.

   If this is the case, we say that z and z' can be concatenated by a *monotone N-chain.* Also, $\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{mon}}(a) \mid a \in H\}$ is called the *monotone catenary degree* of H.

DEFINITION 1.2.5. Let H be an atomic monoid. For $a \in H$ and $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a, x)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.

For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define

$$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\, x \in X\},$$

and we define $\mathsf{t}(H) = \mathsf{t}(H, \mathcal{A}(H_{\mathrm{red}}))$. This is called the *tame degree* of H.

DEFINITION 1.2.6. A monoid homomorphism $\theta : H \to B$ is called a *transfer homomorphism* if it has the following properties:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

(T2) If $a \in H$, r, $s \in B$ and $\theta(a) = rs$, then there exist b, $c \in H$ such that $\theta(b) \sim r$, $\theta(c) \sim s$, and $a = bc$.

DEFINITION 1.2.7. Let $\theta : H \to B$ be a transfer homomorphism of atomic monoids and $\bar{\theta} : \mathsf{Z}(H) \to \mathsf{Z}(B)$ the unique homomorphism satisfying $\bar{\theta}(uH^\times) = \theta(u)B^\times$ for all $u \in \mathcal{A}(H)$. We call $\bar{\theta}$ the extension of θ to the factorization monoids.
For $a \in H$, the *catenary degree in the fibers* $\mathsf{c}(a, \theta)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any two factorizations z, $z' \in \mathsf{Z}(a)$ with $\bar{\theta}(z) = \bar{\theta}(z')$ there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, $\bar{\theta}(z_i) = \bar{\theta}(z)$, and $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$; that is, z and z' can be concatenated by an N-chain in the fiber $\mathsf{Z}(a) \cap \bar{\theta}^{-1}((\bar{\theta}(z)))$.

Also, $\mathsf{c}(H, \theta) = \sup\{\mathsf{c}(a, \theta) \mid a \in H\}$ is called the *catenary degree in the fibers* of H.

DEFINITION 1.2.8. Let $\emptyset \ne L \subset \mathbb{N}_0$ be a non-empty subset and H an atomic monoid.

1. A positive integer $d \in \mathbb{N}$ is called a *distance* of L if there exists some $l \in L$ such that $L \cap [l, l + d] = \{l, l + d\}$. We denote by $\triangle(L)$ the *set of distances* of L. Note that $\triangle(L) = \emptyset$ if and only if $\#L \le 1$.

2. We call
$$\triangle(H) = \bigcup_{a \in H} \triangle(\mathsf{L}(a)) \subset \mathbb{N}$$
the *set of distances* of H and $\min \triangle(H)$ the *minimum distance* of H.
3. Let $a \in H$ and $k, l \in \mathbb{N}$ with $k < l$. We call k and l *adjacent* or, equivalently, *adjacent lengths* of a if $[k, l] \cap \mathsf{L}(a) = \{k, l\}$.

In the next lemma we briefly recall some well known results.

LEMMA 1.2.9. *Let H be an atomic monoid. Then*

1. *H is factorial if and only if $\mathsf{c}(H) = 0$ if and only if $\mathsf{t}(H) = 0$.*
2. *If $\mathsf{c}(H) \le 2$, then H is half-factorial, i.e. $\rho(H) = 1$.*
3. *If H is not factorial, then $2 + \sup \triangle(H) \le \mathsf{c}(H)$.*
4. *If $\mathsf{c}(H) = 3$, then $\triangle(H) = \{1\}$.*
5. *$\mathsf{t}(H) \ge \mathsf{c}(H)$.*

PROOF.

1. Follows by [**14**, Theorem 1.6.3.1] and [**14**, Theorem 1.6.6.1].
2. Follows by [**14**, Theorem 1.6.3.3].
3. Follows by [**14**, Theorem 1.6.3.2].
4. Follows by [**14**, Theorem 1.6.3.4].
5. Follows by [**14**, Theorem 1.6.6.2]. □

DEFINITION 1.2.10. Let H be an atomic monoid and $m \in \mathbb{N}$. If $H \ne H^\times$, we call the set
$$\mathcal{V}_m(H) = \bigcup_{L \in \mathcal{L}(H), m \in L} L \quad \text{the } \textit{union of sets of lengths} \text{ of } H,$$
and if $H = H^\times$, we set $\mathcal{V}_m(H) = \{m\}$.

For an integral domain R, we set $R^\bullet = R \setminus \{0\}$ for the commutative, cancellative monoid of non-zero elements of R. Additionally, all notions, which were introduced for monoids, are used for domains, too; for example, we write $\mathcal{A}(R)$ instead of $\mathcal{A}(R^\bullet)$ for the set of atoms.

DEFINITION 1.2.11. Let R be an integral domain and $K = \mathsf{q}(R)$ the quotient field of R.

1. We call $\operatorname{spec}(R)$ the set of all prime ideals of R.
2. We set
$$\mathfrak{X}(R) = \{\mathfrak{p} \in \operatorname{spec}(R) \mid \mathfrak{p} \ne 0 \text{ and } \mathfrak{p} \text{ is minimal}\}$$
for the *set of minimal prime ideals* of R.
3. Let $L \supset K$ be a field extension. We call $b \in L$ *integral* over R if there is a monic polynomial $f \in R[X]$ such that $f(b) = 0$.
4. For non-empty subsets X, $Y \subset K$, we define
$$(Y : X) = (Y :_K X) = \{a \in K \mid aX \subset Y\} \text{ and } X^{-1} = (R : X).$$
We denote by $\mathcal{I}(R)$ the *set of all ideals* of R and we call an ideal $\mathfrak{a} \in \mathcal{I}(R)$ *invertible* if $\mathfrak{a}\mathfrak{a}^{-1} = R$. Then we denote by $\mathcal{I}^*(R)$ the *set of all invertible ideals* of R.

5. We call

$$\mathsf{cl}_L(R) = \{b \in L \mid b \text{ is integral over } R\} \quad \text{the } \textit{integral closure} \text{ of } R \text{ in } L$$

and we set $\bar{R} = \mathsf{cl}_K(R)$ for the integral closure of an integral domain (in its quotient field).

Definition 1.2.12. A one-dimensional noetherian domain R is called *locally half-factorial* if $\mathcal{I}^*(R)$ is half-factorial.
Note that this notion of being locally half-factorial does not coincide with the one defined in [**2**] but it coincides with what is called purely locally half-factorial there.
By [**14**, Theorem 3.7.1], we have $\mathcal{I}^*(R) \cong \coprod_{\mathfrak{p}\in\mathfrak{X}(R)}(R_\mathfrak{p}^\bullet)_{\mathrm{red}}$. Thus $\mathcal{I}^*(R)$ is half-factorial if and only if $(R_\mathfrak{p}^\bullet)_{\mathrm{red}}$ is half-factorial for all $\mathfrak{p} \in \mathfrak{X}(R)$.

We briefly fix the notation concerning sequences over finite abelian groups. Let G be an additively written finite abelian group. For a subset $A \subset G$ and an element $g \in G$, we set $-A = \{-a \mid a \in A\}$ and $A - g = \{a - g \mid a \in A\}$. Let $\mathcal{F}(G)$ be the free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For a sequence $S = g_1 \cdot \ldots \cdot g_l$, we call

$|S| = l$ the *length* of S,
$\sigma(S) = \sum_{i=1}^{l} g_i \in G$ the *sum* of S,
$\mathrm{supp}(S) = \{g_1, \ldots, g_l\} \subset G$ the *support* of S,
$\Sigma(S) = \{\sum_{i\in I} g_i \mid \emptyset \neq I \subset [1,l]\} \subset G$ the *set of subsums* of S, and
$-\Sigma(S) = \{\sum_{i\in I}(-g_i) \mid \emptyset \neq I \subset [1,l]\} = \{-g \mid g \in \Sigma(S)\} \subset G$ the *set of negative subsums* of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no non-trivial zero-sum subsequence, i.e. $0 \notin \Sigma(S)$, and
- a *minimal zero-sum sequence* if $1 \neq S$, $\sigma(S) = 0$, and every subsequence $S' \mid S$ with $1 \leq |S'| < |S|$ is zero-sum free.

For a subset $G_0 \subset G$, we set

$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$ for the *block monoid* over G_0 and
$\mathcal{A}(G_0) = \{S \in \mathcal{F}(G_0) \mid S \text{ minimal zero-sum sequence}\} \subset \mathcal{B}(G_0)$.

Then, in fact, $\mathcal{B}(G_0)$ is an atomic monoid and $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is its set of atoms.

Definition 1.2.13. For a subset of a finite (additive) abelian group $G_0 \subset G$, the *Davenport constant* $\mathsf{D}(G_0) \in \mathbb{N}$ is defined to be the supremum of all lengths of sequences in $\mathcal{A}(G_0)$.

Definition 1.2.14. Let $H \subset D$ be monoids.

1. We call $H \subset D$ *saturated* or, equivalently, *a saturated submonoid* if, for all $a, b \in H$, $a \mid b$ in D already implies that $a \mid b$ in H.
2. If $H \subset D$ is a saturated submonoid, then we set $D/H = \{a\mathsf{q}(H) \mid a \in D\}$ and $[a]_{D/H} = a\mathsf{q}(H)$ and we call $\mathsf{q}(D)/\mathsf{q}(H) = \mathsf{q}(D/H)$ the *class group* of H in D.

DEFINITION 1.2.15. Let G be an additive abelian group, $G_0 \subset G$ a subset, T a monoid, $\iota : T \to G$ a homomorphism, and $\sigma : \mathcal{F}(G_0) \to G$ the unique homomorphism such that $\sigma(g) = g$ for all $g \in G_0$. Then we call

$$\mathcal{B}(G_0, T, \iota) = \{St \in \mathcal{F}(G_0) \times T \mid \sigma(S) + \iota(t) = \mathbf{0}\}$$

the *T-block monoid* over G_0 defined by ι.
If $T = \{1\}$, then $\mathcal{B}(G_0, T, \iota) = \mathcal{B}(G_0)$ is the block monoid of all zero-sum sequences over G_0.

LEMMA 1.2.16. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, let $G = \mathsf{q}(D/H)$ be its class group, let $\iota : T \to G$ be a homomorphism defined by $\iota(t) = [t]_{D/H}$, and suppose each class in G contains some prime element from P. Then*

1. *The map $\beta : H \to \mathcal{B}(G, T, \iota)$, given by $\beta(pt) = [p]_{D/H} + \iota(t) = [p]_{D/H} + [t]_{D/H}$ is a transfer homomorphism onto the T-block monoid over G defined by ι and $\mathsf{c}(H, \beta) \le 2$*
2. *The following inequalities hold:*

$$\begin{aligned} \mathsf{c}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{c}(H) && \le \max\{\mathsf{c}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\}, \\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{c}_{\mathrm{mon}}(H) && \le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\}, \text{ and} \\ \mathsf{t}(\mathcal{B}(G,T,\iota)) \le \quad & \mathsf{t}(H) && \le \mathsf{t}(\mathcal{B}(G,T,\iota)) + \mathsf{D}(G) + 1. \end{aligned}$$

   In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}(G,T,\iota))$ holds if $\mathsf{c}(\mathcal{B}(G,T,\iota)) \ge 2$ and the equality $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota))$ holds if $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \ge 2$.
3. *$\mathcal{L}(H) = \mathcal{L}(\mathcal{B}(G,T,\iota))$, $\triangle(H) = \triangle(\mathcal{B}(G,T,\iota))$, $\min\triangle(H) = \min\triangle(\mathcal{B}(G,T,\iota))$, and $\rho(H) = \rho(\mathcal{B}(G,t,\iota))$.*
4. *We set $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\}$. Then $\mathcal{B}$ and $\mathcal{B}(G,T,\iota)$ have the same arithmetical properties, and*

$$\begin{aligned} \mathsf{c}(\mathcal{B}) \le \quad & \mathsf{c}(H) && \le \max\{\mathsf{c}(\mathcal{B}), \mathsf{c}(H,\beta)\}, \\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \le \quad & \mathsf{c}_{\mathrm{mon}}(H) && \le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}), \mathsf{c}(H,\beta)\}, \text{ and} \\ \mathsf{t}(\mathcal{B}) \le \quad & \mathsf{t}(H) && \le \mathsf{t}(\mathcal{B}) + \mathsf{D}(G) + 1. \end{aligned}$$

   In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B})$ holds if $\mathsf{c}(\mathcal{B}) \ge 2$ and the equality $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B})$ holds if $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \ge 2$.
   Additionally, $\mathcal{L}(H) = \mathcal{L}(\mathcal{B})$, $\triangle(H) = \triangle(\mathcal{B})$, $\min\triangle(H) = \min\triangle(\mathcal{B})$, and $\rho(H) = \rho(\mathcal{B})$.

PROOF.

1. Follows by [**14**, Proposition 3.2.3.3 and Proposition 3.4.8.2].
2. The assertion on the catenary degree follows by [**14**, Theorem 3.2.5.5], the assertion on the monotone catenary degree by [**14**, Lemma 3.2.6], and the assertion on the tame degree by [**14**, Theorem 3.2.5.1].
3. Follows by [**14**, Proposition 3.2.3.5].
4. Since $\mathbf{0} \in \mathcal{B}(G,T,\iota)$ is a prime element, it defines a partition $\mathcal{B}(G,T,\iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\{$. Thus all studied arithmetical invariants coincide for $\mathcal{B}$ and $\mathcal{B}(G,T,\iota)$. Now the assertions follow from part 2 and part 3. ☐

LEMMA 1.2.17. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ its class group, and suppose each class in G contain some $p \in P$.*

1. *If $\#G \geq 3$, then $\min \triangle(H) = 1$, $\rho(H) > 1$, $\mathsf{c}(H) \geq 3$.*
2. *$\rho(H) \leq \mathsf{D}(G)\rho(T)$.*

PROOF. We define a homomorphism $\iota : T \to G$ by $\iota(t) = [t]_{D/H}$ and write $\mathcal{B}(G, T, \iota)$ for the T-block monoid over G defined by ι.

1. Then $\mathcal{B}(G) \subset \mathcal{B}(G, T, \iota)$ is a divisor-closed submonoid. By [**14**, Theorem 6.7.1.2], we have $\min \triangle(G) = 1$, and thus $\min \triangle(\mathcal{B}(G, T, \iota)) = 1$ and $\mathsf{c}(\mathcal{B}(G, T, \iota)) \geq 3$ by Lemma 1.2.9.3. Now the assertions follow by Lemma 1.2.16.2 and Lemma 1.2.16.3.
2. By [**14**, Proposition 3.4.7.5], we have $\rho(\mathcal{B}(G, T, \iota)) \leq \mathsf{D}(G)\rho(T)$. Now the assertion follows again by Lemma 1.2.16.2. $\square$

LEMMA 1.2.18. *Let $D = D_1 \times \ldots \times D_r$ with $r \in \mathbb{N}$ be the product of atomic monoids $D_1, \ldots, D_r$.*
Then

1. *If $\min \triangle(D_1) = 1$, then $\min \triangle(D) = 1$.*
2. *If $\mathcal{V}_m(D_1) = [2, \infty)$ for $m \in \mathbb{N}_{\geq 2}$, then $\mathcal{V}_m(D) = [2, \infty)$.*
3. *$\mathsf{c}(D) = \sup\{\mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\}$.*
4. *$\rho(D) = \sup\{\rho(D_1), \ldots, \rho(D_r)\}$.*
5. *$\triangle(D_1) \cup \ldots \cup \triangle(D_r) \subset \triangle(D)$.*
6. *$\mathsf{t}(D) = \sup\{\mathsf{t}(D_1), \ldots, \mathsf{t}(D_r)\}$.*

PROOF. For $i \in [1, r]$ we have $\mathcal{L}(D_i) \subset \mathcal{L}(D)$, which implies 1, 2, and 5.

3. Follows from [**14**, Proposition 1.6.8.2].
4. Follows from [**14**, Proposition 1.4.5.2].
6. Follows from [**14**, Proposition 1.6.8.4]. $\square$

CHAPTER 2

A characterization of arithmetical invariants by monoids of relations

Since all arithmetical invariants studied in this chapter coincide for a monoid and the associated reduced monoid, we formulate our results only for reduced monoids for the sake of readability.

2.1. On the catenary and the tame degree and the monoid of relations

This section mainly presents all the results from [**22**] with some slight changes in notation in order to be able to formulate some additional results in the next sections easier.

2.1.1. $\mu(H)$.

Definition 2.1.1 ($\mathcal{R}$-relation, cf. [**19**, end of page 3]). Let H be a reduced atomic monoid. Two elements z, $z' \in \mathsf{Z}(H)$ are *$\mathcal{R}$-related* if

- either $z = z' = 1$
- or there exists a finite sequence of factorizations $(z_0, z_1, \dots, z_k)$ such that $z_0 = z$, $z_k = z'$, $\pi_H(z) = \pi_H(z_i)$, and $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$.

We call this sequence an *$\mathcal{R}$-chain* concatenating z and z', and if, additionally, $|z_{i-1}| \leq |z_i|$ for all $i \in [1, k]$, then we call this sequence a *monotone $\mathcal{R}$-chain* concatenating z and z'. If two elements z, $z' \in \mathsf{Z}(H)$ are $\mathcal{R}$-related, we write $z \approx z'$.

Since in our general setting the number of factorizations of an element $a \in H$ is not necessarily finite, the number of different $\mathcal{R}$-equivalence classes of $\mathsf{Z}(a)$ is potentially infinite too.

Definition 2.1.2 ($\mu(a)$, $\mu(H)$, cf. [**19**, first paragraph, page 4]). Let H be a reduced atomic monoid. For $a \in H$, let $\mathcal{R}_a$ denote the *set of $\mathcal{R}$-equivalence classes of* $\mathsf{Z}(a)$ and, for $\rho \in \mathcal{R}_a$, let $|\rho| = \min\{|z| \mid z \in \rho\}$. For $a \in H$, we set

$$\mu(a) = \sup\{|\rho| \mid \rho \in \mathcal{R}_a\} \leq \sup \mathsf{L}(a)$$

and define

$$\mu(H) = \sup\{\mu(a) \mid a \in H, |\mathcal{R}_a| > 1\}.$$

Then $\mu(H) = 0$ if and only if $|\mathcal{R}_a| = 1$ for all $a \in H$.

The following Proposition 2.1.3 is partly based on the second part of the proof of [**6**, Theorem 3.1] and realizes its result in our slightly more general setup.

Proposition 2.1.3. *Let H be a reduced atomic monoid. Then*

$$\mathsf{c}(a) \geq \mu(a) \textit{ for all } a \in H, \textit{ and } \mathsf{c}(H) = \mu(H).$$

PROOF. First we prove

$$\mathsf{c}(a) \geq \mu(a) \text{ for all } a \in H.$$

Let $a \in H$ be such that $|\mathcal{R}_a| > 1$. We may assume that $\mathsf{c}(a) < \infty$. Let $N \in \mathbb{N}_0$ be such that $\mu(a) \geq N$. Let $\rho \in \mathcal{R}_a$ be such that $|\rho| \geq N$ and $z \in \rho$ such that $|z| = |\rho|$. Let $z' \in \mathsf{Z}(a)$ be such that $z \not\approx z'$ and let $z = z_0, z_1, \ldots, z_k = z'$ be a $\mathsf{c}(a)$-chain concatenating z and z'. Let $i \in [1, k]$ be minimal such that $z \not\approx z_i$. Then $z_{i-1} \not\approx z_i$, and therefore

$$N \leq |z_0| \leq |z_{i-1}| \leq \mathsf{d}(z_i, z_{i-1}) \leq \mathsf{c}(a).$$

Till now we have $\mathsf{c}(H) \geq \mu(H)$. Thus it suffices to show

$$\mu(H) \geq \mathsf{c}(H).$$

We show that, for all $N \in \mathbb{N}_0$, all $a \in H$, and all factorizations z, $z' \in \mathsf{Z}(a)$ with $|z| \leq N$ and $|z'| \leq N$, there is a $\mu(H)$-chain from z to z'. We proceed by induction on N. If $N = 0$, then $z = z' = 1$ and $\mathsf{d}(z, z') = 0 \leq \mu(H)$. Suppose $N \geq 1$ and that, for all $a \in H$ and all z, $z' \in \mathsf{Z}(a)$ with $|z| < N$ and $|z'| < N$, there is a $\mu(H)$-chain from z to z'. Now let $a \in H$ and let z, $z' \in \mathsf{Z}(a)$ with $|z| \leq N$ and $|z'| \leq N$. If $z \not\approx z'$, then there are z'', $z''' \in \mathsf{Z}(a)$ such that $z'' \approx z$, $z''' \approx z'$, and z'' and z''' are minimal in their $\mathcal{R}$-classes with respect to their lengths. Since $\gcd(z'', z''') = 1$, we find $\mathsf{d}(z'', z''') = \max\{|z''|, |z'''|\} \leq \mu(a) \leq \mu(H)$. Now it remains to show that, for any two factorizations z, $z' \in \mathsf{Z}(a)$ with $z \approx z'$, $|z| \leq N$, and $|z'| \leq N$, there is a $\mu(H)$-chain concatenating them. By definition, there is an $\mathcal{R}$-chain $z_0, \ldots, z_k$ with $z = z_0$, $z' = z_k$, and $g_i = \gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. By the induction hypothesis, there is a $\mu(H)$-chain from $g_i^{-1} z_{i-1}$ to $g_i^{-1} z_i$ for all $i \in [1, k]$, and thus there is a $\mu(H)$-chain from z_i to z_{i-1} for $i \in [1, k]$; thus there is a $\mu(H)$-chain from z to z'. □

2.1.2. The monoid of relations $\sim_H$.

DEFINITION 2.1.4. Let H be a reduced atomic monoid. We call

$$\sim_H = \{(x, y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi_H(x) = \pi_H(y)\}$$

the *monoid of relations*.

LEMMA 2.1.5. *Let H be a reduced atomic monoid, $\mathcal{P} \subset H$ the set of prime elements of H, and $T = \mathcal{A}(H) \setminus \mathcal{P}$.*

1. *Then $\sim_H = \{(qx, qy) \mid q \in \mathcal{F}(\mathcal{P}), x, y \in \mathcal{F}(T)\}$ and, for all $q \in \mathcal{F}(\mathcal{P})$ and x, $y \in \mathsf{Z}(H)$, we have $(qx, qy) \in \sim_H$ if and only if $(x, y) \in \sim_H$.*
2. *The homomorphism $\varphi : \sim_H \to \mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T)$ defined by $\varphi(qx, qy) = (q, x, y)$ with $z \in \mathcal{F}(\mathcal{P})$, where x, $y \in \mathcal{F}(T)$, is a divisor theory.*
3. *$\sim_H$ is a Krull monoid with class group $\mathsf{q}([T])$, and the set of all classes containing primes is given by $\{v, v^{-1} \mid v \in T\} \cup \{1\}$ if $\mathcal{P} \neq \emptyset$ and by $\{v, v^{-1} \mid v \in T\}$ otherwise. In particular, the set of classes containing primes is finite if and only if T is finite.*

PROOF.

1. Obviously, we have $\mathsf{Z}(H) = \mathcal{F}(\mathcal{P}) \times \mathcal{F}(T)$. Let $(qx, q'y) \in \mathsf{Z}(H) \times \mathsf{Z}(H)$ with q, $q' \in \mathcal{F}(\mathcal{P})$ and x, $y \in \mathcal{F}(T)$. Then $(qx, q'y) \in \sim_H$ if and only if $\pi_H(qx) = \pi_H(q'y)$. Since q, q' are products of prime elements, we find $q = q'$, and thus $\pi_H(x) = \pi_H(y)$.

2. First we show that φ is a divisor homomorphism. Let $(q_1x_1, q_1y_1), (q_2x_2, q_2y_2) \in \sim_H$ be such that $\varphi(q_1x_1, q_1y_1) = (q_1, x_1, y_1) \mid (q_2, x_2, y_2) = \varphi(q_2x_2, q_2y_2)$ in $\mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T)$. Then there exists $(q, x, y) \in \mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T)$ such that $(q_1, x_1, y_1)(q, x, y) = (q_2, x_2, y_2)$. Now we apply π_H and find

$$\begin{aligned}\pi_H(y_1)\pi_H(x) = \pi_H(x_1)\pi_H(x) = \pi_H(x_1x)\\ = \pi_H(x_2) = \pi_H(y_2) = \pi_H(y_1y) = \pi_H(y_1)\pi_H(y).\end{aligned}$$

Thus $\pi_H(x) = \pi_H(y)$, and therefore $(qx, qy) \in \sim_H$ and $(q_1x_1, q_1y_1) \mid_{\sim_H} (q_2x_2, q_2y_2)$. Now we prove that φ is a divisor theory. Since $\mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T) = \mathcal{F}(U)$ with $U = \{(p, 1, 1) \mid p \in \mathcal{P}\} \cup \{(1, t, 1), (1, 1, t) \mid t \in T\}$, we must show that any element of U is the greatest common divisor of the image of a finite subset of $\sim_H$. Let $(p, 1, 1) \in U$. Since $\varphi(p, p) = (p, 1, 1)$, there is nothing to show in this case. Now let $u \in \mathcal{A}(H)$ be not prime such that $(1, u, 1) \in U$. Since $u \in \mathcal{A}(H)$ is not prime, there are $a, b \in H \setminus H^\times$ not divisible by any prime such that $u \mid ab$ but $u \nmid a$ and $u \nmid b$. Now let $z \in \mathsf{Z}(u^{-1}ab)$, $x \in \mathsf{Z}(a)$, and $y \in \mathsf{Z}(b)$ with $u \nmid xy$. Then we find $(1, u, 1) = \gcd(\varphi(zu, xy), \varphi(u, u))$.
3. It is clear by part 2 and [**14**, Theorem 2.4.8.1] that $\sim_H$ is a Krull monoid. Now we compute its class group. We define the map

$$\phi : \begin{cases} \mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T) & \to \quad \mathsf{q}([T]) \\ (q, x, y) & \mapsto \quad \pi_H(x)(\pi_H(y))^{-1}. \end{cases}$$

Obviously, ϕ is a well-defined monoid homomorphism and ϕ is surjective.
By [**14**, Proposition 2.5.1.4], it is sufficient to show that $\phi^{-1}(1) = \varphi(\sim_H)$ in order to prove that the class group of $\sim_H$ equals $\mathsf{q}([T])$. Now let $(q, x, y) \in \mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T)$. Then we find

$$\begin{aligned}\phi(q, x, y) = \pi_H(x)\pi_H(y)^{-1} = 1 \quad &\Leftrightarrow \\ \pi_H(x) = \pi_H(y) \quad &\Leftrightarrow \\ (x, y) \in \sim_H \quad &\Leftrightarrow \\ (qx, qy) \in \sim_H, &\end{aligned}$$

and we are done. For the last part of the proof, we calculate the set of all classes containing prime elements of $\mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T)$. We have $\mathcal{F}(\mathcal{P}) \times \mathcal{F}(T) \times \mathcal{F}(T) = \mathcal{F}(U)$ with $U = \{(p, 1, 1) \mid p \in \mathcal{P}\} \cup \{(1, t, 1), (1, 1, t) \mid t \in T\}$ and find $\{v, v^{-1} \mid v \in T\} \cup \{1\}$ if $\mathcal{P} \neq \emptyset$ and $\{v, v^{-1} \mid v \in T\}$ otherwise. □

As we have seen in the proof of Lemma 2.1.5.2 every element of $\mathsf{Z}(H) \times \mathsf{Z}(H)$ can be written as greatest common divisor of the image of at most two elements from $\sim_H$. In the literature, such a Krull monoid is called a δ_1-semigroup with divisor theory; for reference, see [**27**] and [**28**].

Definition 2.1.6. Let H be a reduced atomic monoid. For $a \in H$, we define

$$\mathcal{A}_a(\sim_H) = \{(x, y) \in \mathcal{A}(\sim_H) \mid \pi_H(x) = a\}.$$

Lemma 2.1.7. *Let H be a reduced atomic monoid and $a \in H$. Then*

1. *$\mathcal{A}(\sim_H) \subset \{(u, u) \mid u \in \mathcal{A}(H)\} \cup \{(x, y) \in \sim_H \mid \gcd(x, y) = 1\}$.*

2. *If $x, y \in \mathsf{Z}(a)$ with $x \not\approx y$, then $(x, y) \in \mathcal{A}_a(\sim_H)$.*

PROOF.

1. Let $(x, y) \in \mathcal{A}(\sim_H)$ and $z = \gcd(x, y)$. If $z = 1$, we are done. Now assume $z \neq 1$. Then $z = u_1 \cdot \ldots \cdot u_k$ for some $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$. Now we find $(x, y) = (z, z)(xz^{-1}, yz^{-1}) = (u_1, u_1) \cdot \ldots \cdot (u_k, u_k)(xz^{-1}, yz^{-1})$. If $k \geq 2$, then $(x, y) \notin \mathcal{A}(\sim_H)$, a contradiction. If $k = 1$, then $(x, y) \in \mathcal{A}(\sim_H)$ implies $(xz^{-1}, yz^{-1}) = (1, 1)$, that is, $x = z = y = u_1 \in \mathcal{A}(H)$.
2. Let $a \in H$ and $x, y \in \mathsf{Z}(a)$ such that $(x, y) \notin \mathcal{A}_a(\sim_H)$. Then, trivially, $(x, y) \notin \mathcal{A}(\sim_H)$ and thus there are $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_H)$ with $k \geq 2$ such that $(x, y) = (x_1, y_1) \cdot \ldots \cdot (x_k, y_k)$. Then $x = x_1 \cdot \ldots \cdot x_k$, $y_1 x_2 \cdot \ldots \cdot x_k$, $y_1 \cdot \ldots \cdot y_k = y$ is an $\mathcal{R}$-chain concatenating x and y, and therefore $x \approx y$. □

DEFINITION 2.1.8. Let H be a reduced atomic monoid and $\sim_H$ its monoid of relations. For $(x, y) \in \sim_H$ and $X \subset \sim_H$, we set

$$\widetilde{\triangle}(x, y) = \big||x| - |y|\big| \text{ and } \widetilde{\triangle}(X) = \{\widetilde{\triangle}(x, y) \mid (x, y) \in X, |x| \neq |y|\}.$$

Now we can prove something like [**6**, Proposition 3.2] for the catenary degree and like in [**25**] for the elasticity, and an additional result for the set of distances.

PROPOSITION 2.1.9. *Let H be a reduced atomic monoid. Then*

1. $\mathsf{c}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_H)\}$;
2. $\rho(H) = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \sim_H \right\} = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_H) \right\}$*; and*
3. $\triangle(H) \subset \widetilde{\triangle}(\sim_H)$, $\min \triangle(H) = \gcd \widetilde{\triangle}(\mathcal{A}(\sim_H)) = \min \widetilde{\triangle}(\sim_H)$, *and* $\max \triangle(H) \leq \max \widetilde{\triangle}(\mathcal{A}(\sim_H))$.

PROOF.

1. Let $a \in H \setminus H^\times$ and let $z, z' \in \mathsf{Z}(a)$ be two different factorizations of a. Then, of course, $(z, z') \in \sim_H$. Thus there are $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_H)$ such that $(z, z') = (x_1, y_1) \cdot \ldots \cdot (x_k, y_k)$. Now we can construct the following chain of factorizations: $z = z_0$ and $z_i = z_{i-1} x_i^{-1} y_i$ for $i \in [1, k]$. Then $z_k = z'$. Since $(x_i, y_i) \in \mathcal{A}(\sim_H)$, we find $\gcd(x_i, y_i) = 1$ or $x_i = y_i = u$ with $u \in \mathcal{A}(H) \subset \mathsf{Z}(H)$ by Lemma 2.1.7.1. This implies that either $\mathsf{d}(z_{i-1}, z_i) = \max\{|x_i|, |y_i|\}$ or $\mathsf{d}(z_{i-1}, z_i) = 0$. Thus z and z' can be concatenated by a $\max\{|x_i|, |y_i| \mid i \in [1, k]\}$-chain. Since $(x, y) \in \mathcal{A}(\sim_H)$ if and only if $(y, x) \in \mathcal{A}(\sim_H)$, the assertion follows.
2. For all $a \in H$, we have that $\mathsf{Z}(a) \times \mathsf{Z}(a) \subset \sim_H$. Thus we find

$$\rho(a) = \frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, x, y \in \mathsf{Z}(a) \right\} = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathsf{Z}(a) \times \mathsf{Z}(a) \cap \sim_H \right\}.$$

The first equality now follows. Since $\mathcal{A}(\sim_H) \subset \sim_H$ is a subset, it is clear that

$$\sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_H) \right\} \leq \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \sim_H \right\}.$$

In order to prove equality, we show the following assertion:

For all $(x, y) \in \sim_H$, there is $(x', y') \in \mathcal{A}(\sim_H)$ such that $\frac{|x'|}{|y'|} \geq \frac{|x|}{|y|}$.

Let $(x, y) \in \sim_H$ and without loss of generality assume $|x| \geq |y|$. Now there is some $n \in \mathbb{N}$ and $(x_i, y_i) \in \mathcal{A}(\sim_H)$ for all $i \in [1, n]$ such that $(x, y) = (x_1, y_1) \cdot \ldots \cdot (x_n, y_n)$.

When we pass to the lengths, we find $|x| = \sum_{i=1}^n |x_i|$ and $|y| = \sum_{i=1}^n |y_i|$. This yields

$$\frac{|x|}{|y|} \cdot |y| = |x| = \sum_{i=1}^n |x_i| = \sum_{i=1}^n \frac{|x_i|}{|y_i|}|y_i| \le \max_{i=1}^n \frac{|x_i|}{|y_i|} \sum_{i=1}^n |y_i| = \max_{i=1}^n \frac{|x_i|}{|y_i|} \cdot |y|.$$

Thus we find

$$\frac{|x|}{|y|} \le \max_{i=1}^n \frac{|x_i|}{|y_i|}.$$

3. Since, for each $d \in \triangle(H)$, there exists $(x,y) \in \mathsf{Z}(H)$ such that $|y| - |x| = d$, the inclusion $\triangle(H) \subset \widetilde{\triangle}(\sim_H)$ is obvious, and therefore $\min \triangle(H) \ge \min \widetilde{\triangle}(\sim_H)$.
Now let $d' \in \widetilde{\triangle}(\sim_H)$. Then there is $(x,y) \in \sim_H$ such that $d' = ||x| - |y||$ by Definition 2.1.8. We may assume $|x| < |y|$. There is some $x' \in \mathsf{Z}(\pi_H(x))$ such that $|x'| \in (|x|, |y|]$ and $|x|$ and $|x'|$ are adjacent lengths of $\pi_H(x)$. Then $d = |x'| - |x| \le |y| - |x| = d'$, and therefore $\min \triangle(H) \le \min \widetilde{(} \sim_H)$ and equality follows.
Now let $d = \max \triangle(H)$. Then there exists $(x,y) \in \sim_H$ and $a \in H$ such that $\pi_H(x) = a$, $|x| - |y| = d$ and $[|y|, |x|] \cap \mathsf{L}(a) = \{|y|, |x|\}$. There are $k \in \mathbb{N}$ and $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_H)$ such that $(x,y) = (x_1,y_1) \cdot \ldots \cdot (x_k, y_k)$. Since $\sum_{i=1}^k |x_i| = |x| > |y| = \sum_{i=1}^k |y_i|$, there exists $j \in [1,k]$ such that $|x_j| > |y_j|$. Now we show $|x_j| - |y_j| \ge d$. We assume to the contrary $|x_j| - |y_j| < d$. We set $z = y_j \prod_{i=1, i\ne j}^k x_i$. Clearly, $z \in \mathsf{Z}(a)$ and $|z| = |x| - (|x_j| - |y_j|) \in [|x| - (d-1), |x| - 1] \cap \mathsf{L}(a)$, a contradiction.
Let now $d' = \gcd \widetilde{\triangle}(\mathcal{A}(\sim_H))$ and $d = \min \widetilde{(} \sim_H)$. Since $d' \mid d$, it remains to prove that $d' \in \widetilde{\triangle}(\sim_H)$. Let $k \in \mathbb{N}$, $(x_1,y_1), \ldots, (x_k,y_k) \in \mathcal{A}(\sim_H)$, and $n_1, \ldots, n_k \in \mathbb{Z}$ be such that

$$d' = \sum_{i=1}^k n_i \, ||x_i| - |y_i|| \, .$$

Replacing (x_i, y_i) by (y_i, x_i) if necessary, we may assume that $n_i i \, ||x_i| - |y_i|| = |n_i|(|x_i| - |y_i|)$ for all $i \in [1,k]$, and we obtain

$$\widetilde{\triangle}\left(\prod_{i=1}^k (x_i, y_i)^{|n_i|}\right) = \widetilde{\triangle}\left(\prod_{i=1}^k x_i^{|n_i|}, \prod_{i=1}^k y_i^{|n_i|}\right) = \left|\left|\prod_{i=1}^k x_i^{|n_i|}\right| - \left|\prod_{i=1}^k y_i^{|n_i|}\right|\right|$$
$$= \left|\prod_{i=1}^k |n_i||x_i| - \left|\prod_{i=1}^k |n_i||y_i|\right| = \left|\sum_{i=1}^k |n_i|(|x_i| - |y_i|)\right| = d' \in \widetilde{\triangle}(\sim_H). \ \square$$

Next, we mimic the ideas from [**6**, page 259 and Theorem 3.2].

Definition 2.1.10. For a reduced atomic monoid H, we set

$$\nu(H) = \sup\{\mu(a) \mid a \in H, \, \mathcal{A}_a(\sim_H) \ne \emptyset, \, |\mathcal{R}_a| > 1\}.$$

Proposition 2.1.11. *Let H be a reduced atomic monoid. Then*

$$\mathsf{c}(H) = \nu(H).$$

Proof. By Proposition 2.1.3, it is sufficient to show that $\mu(H) = \nu(H)$. When we compare the definitions of those two invariants, we see that the only thing we really have to show is that

$$\{a \in H \mid \mathcal{A}_a(\sim_H) \ne \emptyset, \, |\mathcal{R}_a| > 1\} = \{a \in H \mid |\mathcal{R}_a| > 1\}.$$

One inclusion is trivial and, for the other one, let $a \in H$ be such that $|\mathcal{R}_a| > 1$, and let z, $z' \in \mathsf{Z}(a)$ be two factorizations of a such that $z \not\approx z'$ and such that both are minimal in their $\mathcal{R}$-equivalence classes with respect to their lengths. By Lemma 2.1.7.2, we find $(z, z') \in \mathcal{A}_a(\sim_H) \neq \emptyset$. $\square$

Let H be a finitely generated reduced monoid. The following identity will prove crucial for the computations of the catenary degree in Section 2.3.

$$\mathsf{c}(H) = \max\{\mu(a) \mid a \in H,\, \mathcal{A}_a(\sim_H) \neq \emptyset,\, |\mathcal{R}_a| > 1\}. \tag{2.1.1}$$

Theorem 2.1.12. *Let H be a reduced atomic monoid. Then*

1. $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H,\, \mathcal{A}_a(\sim_H) \neq \emptyset\}$.
2. $\mathsf{c}(H) \le \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_H),\, x \not\approx y\}$.

Proof.

1. Obviously, we have $\mathsf{c}(H) \ge \sup\{\mathsf{c}(a) \mid a \in H, \mathcal{A}_a(\sim_H) \neq \emptyset\}$. Since, by Proposition 2.1.3, $\mathsf{c}(a) \ge \mu(a)$ for all $a \in H$, we find by Proposition 2.1.11, that
$$\begin{aligned}\sup\{\mathsf{c}(a) \mid a \in H, \mathcal{A}_a(\sim_H) \neq \emptyset\} &\ge \sup\{\mu(a) \mid a \in H, \mathcal{A}_a(\sim_H) \neq \emptyset\}\\ &\ge \sup\{\mu(a) \mid a \in H, \mathcal{A}_a(\sim_H) \neq \emptyset, |\mathcal{R}_a| > 1\}\\ &= \nu(H) = \mathsf{c}(H).\end{aligned}$$
2. We use Proposition 2.1.11 and find
$$\begin{aligned}\mathsf{c}(H) &= \nu(H)\\ &\le \sup\{\mu(a) \mid a \in H,\, |\mathcal{R}_a| > 1\}\\ &\le \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_H),\, x \not\approx y\}.\end{aligned}$$ $\square$

Definition 2.1.13. Let H be a reduced atomic monoid. For subsets X, $Y \subset \mathsf{Z}(H)$, we set

$$\mathsf{d}(X, Y) = \begin{cases} \min\{\mathsf{d}(x, y) \mid x \in X,\, y \in Y\} & \text{if } X, Y \neq \emptyset,\\ 0 & \text{else}\end{cases}$$

for the *distance* between X and Y. If $X = \{x\}$, we write $\mathsf{d}(\{x\}, Y) = \mathsf{d}(x, Y)$.

Theorem 2.1.14. *Let H be a reduced atomic monoid and $u \in \mathcal{A}(H)$.*

1. $\mathsf{t}(H, u) = \sup\{\mathsf{d}(x, \mathsf{Z}(a) \cap u\mathsf{Z}(H)) \mid a \in uH,\, x \in \mathsf{Z}(a),\, \mathcal{A}_a(\sim_H) \neq \emptyset\}$.
2. $\mathsf{t}(H) = \sup\{\mathsf{d}(x, \mathsf{Z}(a) \cap u\mathsf{Z}(H)) \mid a \in uH,\, x \in \mathsf{Z}(a),\, \mathcal{A}_a(\sim_H) \neq \emptyset,\, u \in \mathcal{A}(H)\}$.
3. $\mathsf{t}(H) \le \sup\{|x| \mid (x, y) \in \mathcal{A}(\sim_H)\}$.

Proof.

1. Let $t = \mathsf{t}(H, u)$ and $d = \sup\{\mathsf{d}(x, \mathsf{Z}(a) \cap u\mathsf{Z}(H)) \mid a \in uH,\, x \in \mathsf{Z}(a),\, \mathcal{A}_a(\sim_H) \neq \emptyset\}$. We first prove that $t \le d$. Assume $a \in uH$. Now we must show that, for all $z \in \mathsf{Z}(a)$, there exists $z' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le d$. Let $z \in \mathsf{Z}(a)$. If $u \mid z$, then we are done by setting $z' = z$, since then $\mathsf{d}(z, z') = 0 \le d$. Now assume that $u \nmid z$. As $a \in uH$, we have $u^{-1}a \in H$, and therefore there is some $\overline{z} \in \mathsf{Z}(u^{-1}a)$. Then $u\overline{z} \in \mathsf{Z}(a)$ and $u \mid u\overline{z}$. Since $(z, u\overline{z}) \in \sim_H$, there exist $n \in \mathbb{N}$ and $(x_1, y_1), \ldots, (x_n, y_n) \in \mathcal{A}(\sim_H)$ such that $(z, u\overline{z}) = (x_1, y_1) \cdot \ldots \cdot (x_n, y_n)$. This implies that $(x_i, y_i) \mid (z, u\overline{z})$ in $\sim_H$ for all $i \in [1, n]$ and that there exists some

$j \in [1, n]$ such that $u \mid y_j$. Observe that $x_j \mid z$ implies that $u \nmid x_j$. Then $(x_j, y_j) \in \mathcal{A}_{\pi_H(x_j)}(\sim_H)$, $\pi_H(x_j) = \pi_H(y_j) \in uH$, and $y_j \in \mathsf{Z}(\pi_H(x_j)) \cap u\mathsf{Z}(H)$. Now take $y' \in \mathsf{Z}(\pi_H(x_j)) \cap u\mathsf{Z}(H)$ such that $\mathsf{d}(x_j, y') = \mathsf{d}(x_j, \mathsf{Z}(\pi_H(x_j)) \cap u\mathsf{Z}(H))$. If we now choose $z' = y'zx_j^{-1}$, then $z' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$, and $\mathsf{d}(z, z') = \mathsf{d}(x_j(zx_j^{-1}), y'zx_j^{-1}) = \mathsf{d}(x_j, y') \le d$. This proves $t \le d$.
Proof of $t \ge d$. Let $N \in \mathbb{N}_0$ be such that $N \le d$. Then there exist $a \in uH$, $x \in \mathsf{Z}(a)$ such that $\mathcal{A}_a(\sim_H) \ne \emptyset$, and $y \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ such that $\mathsf{d}(x, y) = \mathsf{d}(x, \mathsf{Z}(a) \cap u\mathsf{Z}(H)) \ge N$. Let $x \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ be such that $\mathsf{d}(x, z) \le t$. Then $N \le \mathsf{d}(x, y) \le \mathsf{d}(x, z) \le t$. Hence $d \le t$.
2. Obvious by part 1 and the very definition of the tame degree.
3. Shown in [**3**, Proposition 5.2.2]. $\square$

Let H be a reduced atomic monoid. Suppose we have a decomposition $\mathcal{A}(H_{\mathrm{red}}) = \biguplus_{i \in I} A_i$, where I is an index set and the $A_i \subset \mathcal{A}(H_{\mathrm{red}})$ for $i \in I$ are non-empty subsets such that

$$\mathcal{A}(\sim_H) \cap (\mathcal{F}(A_i) \times \mathcal{F}(A_i)) = \{(a, a) \mid a \in A_i\} \text{ for all } i \in I. \tag{2.1.2}$$

Let a, $b \in \mathcal{A}(H_{\mathrm{red}})$ and define an equivalence relation $\simeq$ on $\mathcal{A}(H_{\mathrm{red}})$ by $a \simeq b$ if a, $b \in A_i$ for some $i \in I$. We can extend the canonical projection $\pi_\simeq : \mathcal{A}(H_{\mathrm{red}}) \to \mathcal{A}(H_{\mathrm{red}})/\simeq$ to a monoid epimorphism $\overline{\pi}_\simeq : H_{\mathrm{red}} \to \overline{H} := [[a_i]_\simeq \mid i \in I]$ (well defined by (2.1.2)) onto a reduced, atomic monoid, where $a_i \in A_i$ for all $i \in I$. Of course, the possibly most interesting special case is when I is finite, that is, $\overline{H}$ is a finitely generated, reduced, atomic monoid.
Now we can prove the following result.

THEOREM 2.1.15. *Let H and $\overline{H}$ be as above. Then*

$$\mathsf{c}(\overline{H}) \le \mathsf{c}(H),$$

and, if additionally $\pi_\simeq$ induces a homomorphism from $\sim_H$ onto $\sim_{\overline{H}}$, then

1. *$\mathsf{c}(H) \le \max\{|x| \mid (x, y) \in \mathcal{A}(\sim_{\overline{H}})\}$;*
   in particular, if $\mathsf{c}(\overline{H}) = \max\{|x| \mid (x, y) \in \mathcal{A}(\sim_{\overline{H}})\}$, then $\mathsf{c}(H) = \mathsf{c}(\overline{H})$;
2. *$\rho(H) = \rho(\overline{H}) = \max\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_{\overline{H}}) \right\}$; and*
3. *$\mathsf{t}(\overline{H}) \le \mathsf{t}(H)$.*

PROOF. Since $\pi_\simeq$ is defined as a map from $\mathcal{A}(H_{\mathrm{red}})$ onto $\mathcal{A}(\overline{H})$, it trivially extends to $\pi_\simeq : \mathsf{Z}(H) \to \mathsf{Z}(\overline{H})$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Z}(H) \arrow[r, "\pi_\simeq"] \arrow[d, "\pi_H"] & \mathsf{Z}(\overline{H}) \arrow[d, "\pi_{\overline{H}}"] \\ H_{\mathrm{red}} \arrow[r, "\overline{\pi}_\simeq"] & \overline{H} \end{tikzcd}$$

Now we prove the following two statements.

A1 For all z, $z' \in \mathsf{Z}(H)$, $z \approx z'$ implies $\pi_\simeq(z) \approx \pi_\simeq(z')$.
A2 For all $z \in \mathsf{Z}(H)$, $|z| = |\pi_\simeq(z)|$.

PROOF OF A1. Let z, $z' \in \mathsf{Z}(H)$ be two factorizations such that $\gcd(z, z') \neq 1$. We have $1 \neq \pi_{\simeq}(\gcd(z, z')) \mid \gcd(\pi_{\simeq}(z), \pi_{\simeq}(z'))$ and find $\gcd(\pi_{\simeq}(z), \pi_{\simeq}(z')) \neq 1$. Now the assertion is obvious. □

PROOF OF A2. It is obvious that $|z| = |\pi_{\simeq}(z)|$ for all $z \in \mathsf{Z}(H)$. □

By **A1**, we find $\mu(H) \geq \mu(\overline{H})$, and thus, by Proposition 2.1.3, we have $\mathsf{c}(\overline{H}) = \mu(\overline{H}) \leq \mu(H) = \mathsf{c}(H)$. Now we assume that $\pi_{\simeq}$ induces a homomorphism from $\sim_H$ onto $\sim_{\overline{H}}$.

1. By **A2**, we find $\max\{|x| \mid (x, y) \in \mathcal{A}(\sim_H)\} = \max\{|x| \mid (x, y) \in \mathcal{A}(\sim_{\overline{H}})\}$, whence Proposition 2.1.9 implies that $\mathsf{c}(H) \leq \max\{|x| \mid (x, y) \in \mathcal{A}(\sim_H)\} = \max\{|x| \mid (x, y) \in \mathcal{A}(\sim_{\overline{H}})\}$.
2. Since $\overline{H}$ is finitely generated, $\sim_{\overline{H}}$ is also finitely generated. Thus we have, by **A2**,

$$\sup\left\{\frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_H)\right\} = \sup\left\{\frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_{\overline{H}})\right\} = \max\left\{\frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}(\sim_{\overline{H}})\right\}.$$

   Now everything follows by Proposition 2.1.9.2.
3. Obviously, we have $\mathsf{d}(z, z') \geq \mathsf{d}(\pi_{\simeq}(z), \pi_{\simeq}(z'))$ for all z, $z' \in \mathsf{Z}(H)$. Thus we find $\mathsf{t}(H) \geq \mathsf{t}(\overline{H})$ by Definition 1.2.5. □

2.2. The monotone catenary degree

For the description and computation of the monotone catenary degree, we follow the same two step procedure as in [**3**]. In order to formulate this precisely, we need some definitions.

DEFINITION 2.2.1. Let H be a reduced atomic monoid.

1. For $a \in H$, the *equal catenary degree* $\mathsf{c}_{\mathrm{eq}}(a)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

   For any two factorizations z, $z' \in \mathsf{Z}(a)$ with $|z| = |z'|$, there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, $\mathsf{d}(z_{i-1}, z_i) \leq N$, and $|z_i| = |z|$ for all $i \in [1, k]$.

   If this is the case, we say that z and z' can be concatenated by an *N-equal-length-chain.*

   Also, $\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(a) \mid a \in H\}$ is called the *equal catenary degree* of H.
2. For $a \in H$, we define

$$\mathsf{c}_{\mathrm{ad}}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \mid k, l \in \mathsf{L}(a) \text{ are adjacent}\}$$

   as the *adjacent catenary degree* of a.

   Also, $\mathsf{c}_{\mathrm{ad}}(H) = \sup\{\mathsf{c}_{\mathrm{ad}}(a) \mid a \in H\}$ is called the *adjacent catenary degree* of H.

Then we find

$$\mathsf{c}(H) \leq \mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H),\, \mathsf{c}_{\mathrm{ad}}(H)\}$$

by [**3**, (4.1)]. There exists an example that the tameness of a monoid does not imply the finiteness of the equal catenary degree and therefore not the finiteness of the monotone catenary degree; see [**10**, Example 4.5]. Unfortunately, there are no results about the finiteness of the adjacent catenary degree in tame monoids. In subsection 2.2.4, we will give a variant of the adjacent catenary degree, which can be shown to be finite in tame monoids.

2.2.1. $\mu_{\mathrm{eq}}(H)$ and the monoid of equal-length relations $\sim_{H,\mathrm{eq}}$.
Here we follow the same strategy as in subsection 2.1.1 for the definition of the $\mathcal{R}$-relation and the μ-invariant.

DEFINITION 2.2.2. Let H be a reduced atomic monoid.

1. Two elements $z,\, z' \in \mathsf{Z}(H)$ with $|z| = |z'|$ are $\mathcal{R}_{\mathrm{eq}}$*-related* if
   - either $z = z' = 1$
   - or there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ such that $z_0 = z$, $z_k = z'$, $\pi_H(z) = \pi_H(z_i)$, $\gcd(z_{i-1}, z_i) \neq 1$, and $|z_i| = |z|$ for all $i \in [1, k]$.

   We call this sequence an $\mathcal{R}_{\mathrm{eq}}$*-chain* concatenating z and z'. If two elements $z,\, z' \in \mathsf{Z}(H)$ are $\mathcal{R}_{\mathrm{eq}}$-related, we write $z \approx_{\mathrm{eq}} z'$.
   Obviously, $z \approx_{\mathrm{eq}} z'$ implies $|z| = |z'|$.
2. For $a \in H$ and $k \in \mathsf{L}(a)$, let $\mathcal{R}_{a,k}$ denote the set of $\mathcal{R}_{\mathrm{eq}}$-*equivalence classes* of $\mathsf{Z}_k(a)$. For $a \in H$, we set
$$\mu_{\mathrm{eq}}(a) = \sup\{k \in \mathsf{L}(a) \mid |\mathcal{R}_{a,k}| > 1\}$$
   and define $\mu_{\mathrm{eq}}(H) = \sup\{\mu_{\mathrm{eq}}(a) \mid a \in H\}$.

Then $\mu_{\mathrm{eq}}(H) = 0$ if and only if $|\mathcal{R}_{a,k}| \le 1$ for all $a \in H$ and $k \in \mathsf{L}(a)$.

LEMMA 2.2.3. *Let H be a reduced atomic monoid and let $x,\, y \in \mathsf{Z}(a)$ with* $\min\{|x|, |y|\} > \mathsf{c}_{\mathrm{mon}}(a)$.
Then there is a monotone $\mathcal{R}$-chain concatenating x and y, thus $x \approx y$; in particular, if $|x| = |y|$, then $x \approx_{\mathrm{eq}} y$.

PROOF. Let $a \in H$ and $x,\, y \in \mathsf{Z}(a)$ be such that $\min\{|x|, |y|\} > \mathsf{c}_{\mathrm{mon}}(a)$. We may assume that $|x| \le |y|$. Then there is a monotone $\mathsf{c}_{\mathrm{mon}}(a)$-chain concatenating x and y, say $z_0 = x, z_1, \ldots, z_k = y$. Since, for all $i \in [1, k]$, we have $\mathsf{d}(z_{i-1}, z_i) \le \mathsf{c}_{\mathrm{mon}}(a) < |x| = |z_0|$, we have $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. Thus $z_0, \ldots, z_k$ is a monotone $\mathcal{R}$-chain concatenating x and y, and therefore $x \approx y$. If $|x| = |y|$, then $z_0, \ldots, z_k$ is an equal-length chain, and therefore $x \approx_{\mathrm{eq}} y$. □

THEOREM 2.2.4. *Let H be a reduced atomic monoid. Then*
$$\mathsf{c}_{\mathrm{eq}}(a) \ge \mu_{\mathrm{eq}}(a) \text{ for all } a \in H, \text{ and } \mathsf{c}_{\mathrm{eq}}(H) = \mu_{\mathrm{eq}}(H).$$

PROOF. First we prove $\mathsf{c}_{\mathrm{eq}}(a) \ge \mu_{\mathrm{eq}}(a)$ for all $a \in H$. We may assume that $\mathsf{c}_{\mathrm{eq}}(a) < \infty$ and $\mu_{\mathrm{eq}}(a) \ge 1$. Let $N \in \mathbb{N}$ be such that $N \le \mu_{\mathrm{eq}}(a)$. Then there exists $k \in \mathsf{L}(a)$ such that $|\mathcal{R}_{a,k}| > 1$ and $k \ge N$. Let $z,\, z' \in \mathsf{Z}_k(a)$ be such that $z \not\approx_{\mathrm{eq}} z'$, and let $z = z_0, z_1, \ldots, z_n = z'$ be a $\mathsf{c}_{\mathrm{eq}}(a)$-equal-length chain concatenating z and z'. Now we choose $i \in [0, n-1]$ minimal such that $z \not\approx_{\mathrm{eq}} z_i$. Then $z_{i-1} \not\approx_{\mathrm{eq}} z_i$, and we find
$$\mathsf{c}_{\mathrm{eq}}(a) \ge \mathsf{d}(z_{i-1}, z_i) = k \ge N.$$
Now we prove $\mu_{\mathrm{eq}}(H) \ge \mathsf{c}_{\mathrm{eq}}(H)$. We show that, for all $N \in \mathbb{N}_0$, all $a \in H$, and all factorizations $z,\, z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$, there is a $\mu_{\mathrm{eq}}(H)$-equal-length-chain from z to z'. We proceed by induction on N. If $N = 0$, then $z = z' = 1$ and $\mathsf{d}(z, z') = 0 \le \mu_{\mathrm{eq}}(H)$. Suppose $N \ge 1$ and that, for all $a \in H$ and all $z,\, z' \in \mathsf{Z}(a)$ with $|z| = |z'| < N$, there

is a $\mu_{\mathrm{eq}}(H)$-equal-length-chain from z to z'. Now let $a \in H$ and let $z, z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$. If $z \not\approx_{\mathrm{eq}} z'$, then $\mu_{\mathrm{eq}}(H) \ge \mu_{\mathrm{eq}}(a) \ge |z| = \mathsf{d}(z, z')$. Now it remains to show that, for any two factorizations $z, z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$ and $z \approx_{\mathrm{eq}} z'$, there is a $\mu_{\mathrm{eq}}(H)$-equal-length-chain concatenating them. By definition, there is an $\mathcal{R}_{\mathrm{eq}}$-chain $z_0, \ldots, z_k$ with $z_0 = z$, $z' = z_k$, $g_i = \gcd(z_{i-1}, z_i) \ne 1$, and $|z_i| = |z|$ for all $i \in [1, k]$. By induction hypothesis, there is a $\mu_{\mathrm{eq}}(H)$-equal-length-chain from $g_i^{-1} z_{i-1}$ to $g_i^{-1} z_i$ for all $i \in [1, k]$, and thus there is a $\mu_{\mathrm{eq}}(H)$-equal-length-chain from z_{i-1} to z_i for $i \in [1, k]$; thus there is a $\mu_{\mathrm{eq}}(H)$-equal-length chain from z to z'. $\square$

Definition 2.2.5. Let H be a reduced atomic monoid.

$$\sim_{H,\mathrm{eq}} = \{(x, y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi(x) = \pi(y) \text{ and } |x| = |y|\} = \{(x, y) \in \sim_H \mid |x| = |y|\}$$

is called the *monoid of equal-length relations* of H.

By [**3**, Proposition 4.4.1], $\sim_{H,\mathrm{eq}} \subset \sim_H$ is a saturated submonoid and hence a Krull monoid, and, by [**3**, Proposition 4.4.2], $\sim_{H,\mathrm{eq}}$ is finitely generated if H_{red} is finitely generated.

Definition 2.2.6. Let H be a reduced atomic monoid. For $a \in H$, we set

$$\mathcal{A}_a(\sim_{H,\mathrm{eq}}) = \{(x, y) \in \mathcal{A}(\sim_{H,\mathrm{eq}}) \mid \pi(x) = aH^\times\}.$$

Lemma 2.2.7. *Let H be a reduced atomic monoid, $a \in H$, and $z, z' \in \mathsf{Z}(a)$ such that $z \not\approx_{\mathrm{eq}} z'$.*
Then $(z, z') \in \mathcal{A}_a(\sim_{H,\mathrm{eq}})$.

Proof. Let $a \in H$ and $z, z' \in \mathsf{Z}(a)$ be such that $(z, z') \notin \mathcal{A}_a(\sim_{H,\mathrm{eq}})$. Then, trivially, $(z, z') \notin \mathcal{A}(\sim_{H,\mathrm{eq}})$ and thus there are $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_{H,\mathrm{eq}})$ with $k \ge 2$ such that $(z, z') = (x_1, y_1) \cdot \ldots \cdot (x_k, y_k)$. Then $z = x_1 \cdot \ldots \cdot x_k$, $y_1 x_2 \cdot \ldots \cdot x_k$, $y_1 \cdot \ldots \cdot y_k = z'$ is an $\mathcal{R}_{\mathrm{eq}}$-chain concatenating z and z', and therefore $z \approx_{\mathrm{eq}} z'$. $\square$

Theorem 2.2.8. *Let H be a reduced atomic monoid. Then*

$$\begin{aligned}\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mu_{\mathrm{eq}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset,\, |\mathcal{R}_{a,k}| > 1 \textit{ for some } k \in \mathsf{L}(a)\}\\ = \sup\{k \in \mathbb{N} \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset,\, k \in \mathsf{L}(a),\, |\mathcal{R}_{a,k}| > 1\}.\end{aligned}$$

Proof. By Theorem 2.2.4, we have $\mathsf{c}_{\mathrm{eq}}(H) = \mu_{\mathrm{eq}}(H)$ and, by Definition 2.2.2.2, the second and the third equality are obvious. Thus it suffices to show that

$$\begin{aligned}\{\mu_{\mathrm{eq}}(a) \mid a \in H,\, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\} =\\ \{\mu_{\mathrm{eq}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset,\, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\}.\end{aligned}$$

The inclusion from right to left is clear. Now let $a \in H$ and $k \in \mathsf{L}(a)$ be such that $|\mathcal{R}_{a,k}| > 1$. Then there exist $z, z' \in \mathsf{Z}_k(a)$ such that $z \not\approx_{\mathrm{eq}} z'$. By Lemma 2.2.7, we find $(z, z') \in \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset$. $\square$

2.2.2. $\mu_{\mathrm{ad}}(H)$ and the monoid of monotone relations $\sim_{H,\mathrm{mon}}$.

In principal, we follow again the same strategy as in subsection 2.1.1 and 2.2.1 for the μ-invariant and the μ_{eq}-invariant. But here we cannot construct an equivalence relation like the $\mathcal{R}$-relation or the $\mathcal{R}_{\mathrm{eq}}$-relation.

LEMMA 2.2.9. *Let H be a reduced atomic monoid, $a \in H$, and $k, l \in \mathsf{L}(a)$. Then*

$$\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = \max\{k, l\} \quad \Longleftrightarrow \quad \gcd(x, y) = 1 \textit{ for all } x \in \mathsf{Z}_k(a) \textit{ and } y \in \mathsf{Z}_l(a).$$

PROOF. The result follows immediately by Definition 2.1.13. □

DEFINITION 2.2.10. Let H be a reduced atomic monoid. For $a \in H$, we set

$$\mu_{\mathrm{ad}}(a) = \sup\{k \in \mathsf{L}(a) \mid \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = k \text{ for } l \in \mathsf{L}(a),\, l < k,\, l \text{ adjacent to } k\}.$$

Then we set $\mu_{\mathrm{ad}}(H) = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H\}$.

LEMMA 2.2.11. *Let H be a reduced atomic monoid, $a \in H$, $k, l \in \mathsf{L}(a)$ adjacent with $k < l$, $x \in \mathsf{Z}_k(a)$, and $y \in \mathsf{Z}_l(a)$ such that there is a monotone $\mathcal{R}$-chain from x to y. Then $\mu_{\mathrm{ad}}(a) \neq l$.*

PROOF. Let $a \in H$, $k, l \in \mathsf{L}(a)$ adjacent with $k < l$, $x \in \mathsf{Z}_k(a)$, and $y \in \mathsf{Z}_l(a)$ be such that there is a monotone $\mathcal{R}$-chain from x to y, say $z_0 = x, z_1, \ldots, z_n = y$ for some $n \in \mathbb{N}$. Now choose $i \in [1, n]$ minimal such that $|z_i| = l$. Due to the minimality of i, we find $z_{i-1} \in \mathsf{Z}_k(a)$. Since $\gcd(z_{i-1}, z_i) \neq 1$, we find $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) < l$, and therefore $\mu_{\mathrm{ad}}(a) \neq l$. □

THEOREM 2.2.12. *Let H be an atomic monoid. Then*

$$\mathsf{c}_{\mathrm{ad}}(a) \geq \mu_{\mathrm{ad}}(a) \textit{ for all } a \in H \textit{ and } \mathsf{c}_{\mathrm{ad}}(H) = \mu_{\mathrm{ad}}(H).$$

PROOF. First let $a \in H$. We show that $\mathsf{c}_{\mathrm{ad}}(a) \geq \mu_{\mathrm{ad}}(a)$, and then will $\mathsf{c}_{\mathrm{ad}}(H) \geq \mu_{\mathrm{ad}}(H)$ follow by passing to the supremum on both sides. If $\mu_{\mathrm{ad}}(a) = 0$ or $\mu_{\mathrm{ad}}(a) = \infty$, this is trivial. Now let $\mu_{\mathrm{ad}}(a) = k \in \mathbb{N}$. Then there is $l \in \mathsf{L}(a)$ and $l < k$ with l adjacent to k. Then, by Definition 2.2.1.2, $\mathsf{c}_{\mathrm{ad}}(a) \geq \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = \max\{k, l\} = k = \mu_{\mathrm{ad}}(a)$.
Now we prove $\mu_{\mathrm{ad}}(H) \geq \mathsf{c}_{\mathrm{ad}}(H)$. We must prove that $\mathsf{c}_{\mathrm{ad}}(a) \leq \mu_{\mathrm{ad}}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathsf{c}_{\mathrm{ad}}(a) > \mu_{\mathrm{ad}}(H)$. Let $k \in \mathbb{N}$ be minimal such that there is some $l < k$ and $a \in H$ such that k and l are adjacent lengths of a and $\mathsf{c}_{\mathrm{ad}}(a) = \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a))$. If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) < k$, then there are some $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$ such that $g = \gcd(x, y) \neq 1$. If $b = \pi_H(g^{-1}x)$, then $k - |g|$ and $l - |g|$ are adjacent lengths of b and

$$\mathsf{c}_{\mathrm{ad}}(a) = \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \leq \mathsf{d}(\mathsf{Z}_{k-|g|}(b), \mathsf{Z}_{l-|g|}(b)) \leq \mathsf{c}_{\mathrm{ad}}(b),$$

and by the minimal choice of k we infer that $\mathsf{c}_{\mathrm{ad}}(b) \leq \mu_{\mathrm{ad}}(H)$, a contradiction. □

DEFINITION 2.2.13. Let H be an atomic monoid and $\sim_H$ the monoid of relations of H. Then we set

$$\sim_{H,\mathrm{mon}} = \{(x, y) \in \sim_H \mid |x| \leq |y|\} \text{ for the } \textit{monoid of monotone relations} \text{ of } H.$$

Unfortunately, $\sim_{H,\mathrm{mon}} \subset \sim_H$ is not saturated.

DEFINITION 2.2.14. Let H be a reduced atomic monoid. For $a \in H$, we set

$$\mathcal{A}_a(\sim_{H,\mathrm{mon}}) = \{(x,y) \in \mathcal{A}(\sim_{H,\mathrm{mon}}) \mid \pi(x) = aH^\times\}.$$

LEMMA 2.2.15. *Let H be a reduced atomic monoid, $a \in H$, and let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$.*
If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = l$, then $(x,y) \in \mathcal{A}_a(\sim_{H,\mathrm{mon}})$ for all $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$.

PROOF. Let $a \in H$, let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$ and $\mathsf{d}(Z_k(a), \mathsf{d}(Z_l(a)) = l$, and let $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$. Now suppose $(x,y) \notin \mathcal{A}_a(\sim_{H,\mathrm{mon}})$. Then, trivially, $(x,y) \notin \mathcal{A}(\sim_{H,\mathrm{mon}})$ and there are $(x_1,y_1), \ldots, (x_k,y_k) \in \mathcal{A}(\sim_{H,\mathrm{mon}})$ with $k \geq 2$ and $|y_1| - |x_1| \leq \ldots \leq |y_k| - |x_k|$. Then we set $x' = x_1^{-1} y_1 x$. If $|y_1| - |x_1| = 0$, we find $|x'| = k$ and $\gcd(x', y) \neq 1$, a contradiction to $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = l$. Otherwise, if $|y_1| - |x_1| > 0$, then $k = |x| < |x'| < |y| = l$, a contradiction to k and l adjacent. ☐

THEOREM 2.2.16. *Let H be a reduced atomic monoid. Then*

$$\mathsf{c}_{\mathrm{ad}}(H) = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \neq \emptyset\}.$$

PROOF. By Theorem 2.2.12 and Definition 2.2.10, we find

$$\mathsf{c}_{\mathrm{ad}}(H) = \mu_{\mathrm{ad}}(H) = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H\}.$$

Thus it suffices to show that

$$\sup\{\mu_{\mathrm{ad}}(a) \mid a \in H\} = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{mon}})\}.$$

In fact, we only have to show that $\sup\{\mu_{\mathrm{ad}}(a) \mid a \in H\} \leq \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{mon}})\}$. Now let $a \in H$ and $\mu_{\mathrm{ad}}(a) = k \in \mathbb{N}$. Then there is $l \in \mathsf{L}(a)$ with $l < k$, l adjacent to k, and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = k$. Now let $x \in \mathsf{Z}_l(a)$ and $y \in \mathsf{Z}_k(a)$. Then we have $\gcd(x,y) = 1$. By Lemma 2.2.15, we have $(x,y) \in \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \neq \emptyset$. ☐

2.2.3. The monotone catenary degree and some special situations.

COROLLARY 2.2.17. *Let H be a reduced atomic monoid. Then*

$$\mathsf{c}_{\mathrm{mon}}(H) = \sup(\{\mu_{\mathrm{eq}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{eq}}),\, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\} \cup \{\mu_{\mathrm{ad}}(a) \mid a \in H,\, \mathcal{A}_a(\sim_{H,\mathrm{mon}})\}).$$

PROOF. The result follows immediately by Theorem 2.2.8 and Theorem 2.2.16. ☐

LEMMA 2.2.18. *Let H be a reduced atomic monoid. Then*

1. $\mathsf{c}_{\mathrm{eq}}(H) \leq \sup\{|y| \mid (x,y) \in \mathcal{A}(\sim_{H,\mathrm{eq}}),\, x \not\approx_{\mathrm{eq}} y\}$.
2. $\mathsf{c}_{\mathrm{ad}}(H) \leq \sup\{|y| \mid (x,y) \in \mathcal{A}(\sim_{H,\mathrm{mon}}),\, |x| < |y|,\, |x|, |y| \in \mathsf{L}(\pi_H(x))$ *adjacent, and there is no monotone $\mathcal{R}$-chain from x to y*$\}$.
3. $\mathsf{c}_{\mathrm{mon}}(H) \leq \sup\{|y| \mid (x,y) \in \mathcal{A}(\sim_{H,\mathrm{mon}})$, *there is no monotone $\mathcal{R}$-chain from x to y, and either $|x| = |y|$ or $|x|, |y| \in \mathsf{L}(\pi_H(x))$ are adjacent*$\}$.

PROOF.

1. The inequality $\mathsf{c}_{\mathrm{eq}}(H) \leq \sup\{|y| \mid (x,y) \in \mathcal{A}(\sim_{H,\mathrm{eq}})$ has been proven in [**3**, Proposition 4.4.3]. The slightly stronger statement here, follows immediately by the definition of $\mu_{\mathrm{eq}}(\cdot)$; see Definition 2.2.2.2.

2. By Theorem 2.2.16, we have $\mathsf{c}_{\mathrm{ad}}(H) \le \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \ne \emptyset\}$. Now the assertion follows from Lemma 2.2.11, Lemma 2.2.15, and the definition of $\mu_{\mathrm{ad}}(\cdot)$, see Definition 2.2.10.
3. The assertion now follows from

$$\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H), \mathsf{c}_{\mathrm{ad}}(H)\}, \text{ and } \mathcal{A}(\sim_{H,\mathrm{eq}}) \subset \mathcal{A}(\sim_{H,\mathrm{mon}}). \qquad \square$$

LEMMA 2.2.19. *Let H be a reduced atomic monoid.*

1. *If H is half-factorial, then $\mathsf{c}_{\mathrm{ad}}(H) = 0$ and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{eq}}(H) = \mathsf{c}(H)$.*
2. *If $a \in H$ satisfies $|\mathsf{L}(a)| \le 2$, then $\mu_{\mathrm{ad}}(a) \le \mathsf{t}(H)$.*

PROOF.

1. Since, for all $a \in H$, $|\mathsf{L}(a)| = 1$, we have no adjacent lengths, it follows that $\mathsf{c}_{\mathrm{ad}}(H) = 0$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{eq}}(H)$. As—in this special situation—every chain of factorizations is an equal-length chain of factorizations, we get $\mathsf{c}_{\mathrm{eq}}(H) = \mathsf{c}(H)$.
2. Let $a \in H$ e such that $|\mathsf{L}(a)| \le 2$. If $|\mathsf{L}(a)| = 1$, then $\mu_{\mathrm{ad}}(a) = 0$. Now suppose $|\mathsf{L}(a)| = 2$. If $\mu_{\mathrm{ad}}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\mathrm{ad}}(a) > 0$. Then $\mu_{\mathrm{ad}}(a) = \max \mathsf{L}(a)$, and thus $\gcd(x, y) = 1$ for all $x, y \in \mathsf{Z}(a)$ with $|x| = \min \mathsf{L}(a)$ and $|y| = \max \mathsf{L}(a)$. Let $x, y \in \mathsf{Z}(a)$ with $|x| = \min \mathsf{L}(a)$ and $|y| = \max \mathsf{L}(a)$ and choose $u \in \mathcal{A}(H)$ such that $x \in \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)$. Then there is no $y' \in \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)$ with $|y'| = |y|$. Now we find

$$\mathsf{t}(H) \ge \mathsf{t}(a, uH^\times) \ge \mathsf{d}(y, \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)) = |y| = \max \mathsf{L}(a) = \mu_{\mathrm{ad}}(a). \qquad \square$$

2.2.4. The m-adjacent catenary degree.

Next we formulate another variant of the catenary degree, which is a somewhat similar to the adjacent catenary degree and equals it in a special situation. The main difference is that we can prove that the m-adjacent catenary degree is finite for tame monoids when m is sufficiently large.

DEFINITION 2.2.20. Let H be a reduced atomic monoid.

1. Let $a \in H$ and $M \subset \mathbb{N}$. Then we set

$$\mathsf{Z}_M(a) = \{x \in \mathsf{Z}(a) \mid |x| \in M\}.$$

2. For $a \in H$ and $m \in \mathbb{N}$, we define

$$\mathsf{c}_{\mathrm{ad},m}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a)) \mid k \in \mathsf{L}(a)\}$$

as the *m-adjacent catenary degree* of a.
Also, $\mathsf{c}_{\mathrm{ad},m}(H) = \sup\{\mathsf{c}_{\mathrm{ad},m}(a) \mid a \in H\}$ is called the *m-adjacent catenary degree* of H.

Obviously, we find

$$\mathsf{c}_{\mathrm{ad},m}(H) \begin{cases} = 0 & m < \min \triangle(H) \\ \le \mathsf{c}_{\mathrm{ad}}(H) & \\ = \mathsf{c}_{\mathrm{ad}}(H) & \triangle(H) = \{n\} \text{ and } n \le m < 2n. \end{cases}$$

Definition 2.2.21. Let H be a reduced atomic monoid. For $a \in H$ and $m \in \mathbb{N}$, we set

$$\mu_{\mathrm{ad},m}(a) = \sup\{k \in \mathsf{L}(a) \mid \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a)) = k\}.$$

Then we set $\mu_{\mathrm{ad},m}(H) = \sup\{\mu_{\mathrm{ad},m}(a) \mid a \in H\}$.

Since the definitions of the m-adjacent catenary degree and of $\mu_{\mathrm{ad},m}(H)$ are similar to those of the adjacent catenary degree and $\mu_{\mathrm{ad}}(H)$, we can now prove the analog of Theorem 2.2.12.

Theorem 2.2.22. *Let H be a reduced atomic monoid and $m \in \mathbb{N}$. Then*

$$\mathsf{c}_{\mathrm{ad},m}(a) \ge \mu_{\mathrm{ad},m}(a) \text{ for all } a \in H \text{ and } \mathsf{c}_{\mathrm{ad},m}(H) = \mu_{\mathrm{ad},m}(H).$$

Proof. For $m < \min \triangle(H)$, we have $\mathsf{c}_{\mathrm{ad},m}(H) = 0 = \mu_{\mathrm{ad},m}(H)$ by definition. Now let $m \in \mathbb{N}$ and $m \ge \min \triangle(H)$.
First we let $a \in H$ and show that $\mathsf{c}_{\mathrm{ad},m}(a) \ge \mu_{\mathrm{ad},m}(a)$, after which $\mathsf{c}_{\mathrm{ad},m}(H) \ge \mu_{\mathrm{ad},m}(H)$ follows by passing to the supremum on both sided. If $\mu_{\mathrm{ad},m}(a) = 0$ or $\mu_{\mathrm{ad},m}(a) = \infty$, this is trivial. Now let $\mu_{\mathrm{ad},m}(a) = k \in \mathbb{N}$ and $[k-m,k) \cap \mathsf{L}(a) = \{l_1, \ldots, l_n\}$. Then, by Definition 2.2.20.2, $\mathsf{c}_{\mathrm{ad},m}(a) \ge \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a)) = k = \mu_{\mathrm{ad},m}(a)$.
Now we prove $\mu_{\mathrm{ad},m}(H) \ge \mathsf{c}_{\mathrm{ad},m}(H)$. We must prove that $\mathsf{c}_{\mathrm{ad},m}(a) \le \mu_{\mathrm{ad},m}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathsf{c}_{\mathrm{ad},m}(a) > \mu_{\mathrm{ad},m}(H)$. Let $k \in \mathbb{N}$ be minimal such that there is $a \in H$ with $\mathsf{c}_{\mathrm{ad},m}(a) = \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a))$. If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a)) < k$, then there are some $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_{[k-m,k)}(a)$ such that $g = \gcd(x,y) \ne 1$. If $b = \pi_H(g^{-1}x)$, then $k - |g|$, $|y| - |g| \in \mathsf{L}(b) \cap [k-|g|-m, k-|g|]$ and

$$\mathsf{c}_{\mathrm{ad},m}(a) = \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k)}(a)) \le \mathsf{d}(\mathsf{Z}_{k-|g|}(b), \mathsf{Z}_{[k-|g|-m,k-|g|)}(b)) \le \mathsf{c}_{\mathrm{ad},m}(b),$$

and, by the minimal choice of k, we infer that $\mathsf{c}_{\mathrm{ad},m}(b) \le \mu_{\mathrm{ad},m}(H)$, a contradiction. □

Lemma 2.2.23. *Let H be a reduced atomic monoid and $\mathsf{t}(H) < \infty$. Then*

$$\mathsf{c}_{\mathrm{ad},m}(H) \le \mathsf{t}(H) \quad \text{for all } m \ge \mathsf{t}(H).$$

Proof. Let $m \ge \mathsf{t}(H)$. By Theorem 2.2.22, it suffices to show that $\mu_{\mathrm{ad},m}(a) \le \mathsf{t}(H)$ for all $a \in H$. Let $a \in H$. If $\mu_{\mathrm{ad},m}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\mathrm{ad},m}(a) = k > 0$. Then we have $\mathsf{L}(a) \cap [k-m,k) = \{l_1, \ldots, l_n\}$ and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{l_i}(a)) = k$ for all $i \in [1,n]$. Then $\gcd(x,y) = 1$ for all $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_{l_1}(a)$. Now let $x \in \mathsf{Z}_k(a)$, $y \in \mathsf{Z}_{l_1}(a)$, and choose $u \in \mathcal{A}(H)$ such that $y \in \mathsf{Z}(a) \cap uH^\times\mathsf{Z}(H)$. We find

$$\begin{aligned}(2.2.1)\quad & \mathsf{t}(H) \ge \mathsf{t}(a, uH^\times) \ge \mathsf{d}(x, \mathsf{Z}(a) \cap uH^\times\mathsf{Z}(H)) \\ &= \min\{\mathsf{d}(x, \mathsf{Z}_l(a) \cap uH^\times\mathsf{Z}(H)) \mid l \in \mathsf{L}(a),\, l \ne k\} \ge \min\{k, m+1\} = k = \mu_{\mathrm{ad},m}(a),\end{aligned}$$

since $m + 1 > \mathsf{t}(H)$. □

Another interesting observation arising from the proof of Lemma 2.2.23 is the fact that the crucial inequality (2.2.1) might fail for $m < \mathsf{t}(H)$ for some $a \in H$ (of course with $\mu_{\mathrm{ad},m}(a) > 0$). Unfortunately, Lemma 2.2.23 can never be used to bound $\mathsf{c}_{\mathrm{ad}}(H)$ since $\mathsf{c}_{\mathrm{ad}}(H) = \mathsf{c}_{\mathrm{ad},m}(H)$ for $m = \min \triangle(H)$ if $\#\triangle(H) = 1$, but then $\mathsf{t}(H) \ge m + 2 > m$, and therefore Lemma 2.2.23 does not hold for $\mathsf{c}_{\mathrm{ad}}(H)$.

2.3. An algorithmic approach to the computation of arithmetical invariants

By [**14**, Theorem 3.7.1], the arithmetic of weakly Krull domains, e.g. some (non-principal) order in an algebraic number field or $R = \mathbb{F}_p[X^n, \dots, X^{2n-1}]$ with $p \in \mathbb{P}$ and $n \in \mathbb{N}_{\geq 2}$, can mostly be described by studying appropriate T-block monoids, i.e. $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$. In this section, we exploit the results from [**6**], [**7**], [**22**], and Section 2.1 and Section 2.2, mainly (2.1.1), Theorem 2.1.14.2, and Corollary 2.2.17 together with recent programming techniques—see [**15**] and [**20**]—and parallelization to explicitly compute various arithmetical invariants, namely the elasticity, the catenary degree, the monotone catenary degree, and a bound for the tame degree of the studied domains.

2.3.1. Preliminaries about zero-sum sequences and *T*-block monoids.

In order to be able to describe the set of atoms of a T-block monoid precisely, we use the terminology of sequences over finite abelian groups.

For our algorithmic considerations in the forthcoming sections, it will be very useful to have some sort of order defined on the elements of a finite abelian group G. By the structure theorem for finitely generated abelian groups, there are uniquely determined $r \in \mathbb{N}_0$ and $n_1, \dots, n_r \in \mathbb{N}$ such that there is a group isomorphism $\varphi : G \to \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z}$ and $1 < n_1 \mid \dots \mid n_r$. For $i \in [1, r)$, we choose $[0, n_i)$ as a system of representatives for $\mathbb{Z}/n_i\mathbb{Z}$. Now we can compare two elements $g_1, g_2 \in G$ by comparing $\varphi(g_1)$ and $\varphi(g_2)$ with respect to the lexicographic order. For short, we simply write $g_1 \leq g_2$ respectively $g_1 \geq g_2$.

In particular in subsection 2.3.4, we will need some kind of coordinate representation for the elements of a T-block monoid, i.e. a monoid isomorphism mapping a T-block monoid onto a submonoid of $\mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z}$ for some m, $r \in \mathbb{N}_0$ and $n_1, \dots, n_r \in \mathbb{N}$. Let G be a finite abelian group, T a finitely generated monoid, and $\iota : T \to G$ a homomorphism. Let $T = D_1 \times \dots \times D_r$ be a product of finitely primary monoids $D_i \subset [p_1^{(i)}, \dots, p_{r_i}^{(i)}] \times \widehat{D_i}^\times = \widehat{D_i}$ where $r_i \in \mathbb{N}$, $\widehat{D_i}^\times$ are finitely generated abelian groups for $i \in [1, r]$. Then there are uniquely determined l_i, $k_i \in \mathbb{N}_0$ such that there is an isomorphism $\phi_i : \widehat{D_i}^\times \to \mathbb{Z}^{l_i} \times \mathbb{Z}/n_1^{(i)}\mathbb{Z} \times \dots \times \mathbb{Z}/n_{k_i}^{(i)}$ with $1 < n_1^{(i)} \mid \dots \mid n_{k_i}^{(i)}$ for $i \in [1, r]$. This isomorphism can be extended to an isomorphism $\bar{\phi}_i : \widehat{D_i} \to N_0^{r_i} \times \phi_i(\widehat{D_i}^\times)$ for $i \in [1, r]$. Now there is an isomorphism $\phi = \bar{\phi}_1 \times \dots \times \bar{\phi}_r : \widehat{T} \to \bar{\phi}_1(\widehat{D_1}) \times \dots \times \bar{\phi}_r(\widehat{D_r})$. This again can be extended to an isomorphism $\bar{\varphi} : \mathcal{F}(G) \times \widehat{T} \to N_0^{\#G} \times \phi(\widehat{T})$. Now we can define the desired isomorphism by restriction of $\bar{\varphi}$ to the T-block monoid $\mathcal{B}(G, T, \iota)$ as follows:
(2.3.1)

$$\varphi = \bar{\varphi}|\mathcal{B}(G,T,\iota) : \mathcal{B}(G,T,\iota) \to \bar{\varphi}(\mathcal{B}(G,T,\iota)) \subset \mathbb{N}_0^{\#G} \times \prod_{i=1}^{r} \left(\mathbb{N}_0^{r_i} \times \mathbb{Z}^{l_i} \times \prod_{j=1}^{k_i} \mathbb{Z}/n_j^{(i)}\mathbb{Z} \right).$$

2.3.2. The set of atoms $\mathcal{A}(G)$ of a block monoid.

Based on ideas from [**15**], we give an algorithm for the computation of the set of atoms $\mathcal{A}(G)$ for a finite additive abelian group G. The problem of computing $\mathcal{A}(G)$ grows exponentially in terms of $\#G$, but, for very small groups as the ones involved in subsection 2.3.5, it can be easily performed—sometimes even by hand. Unfortunately, we have to do some sort of brute force search in the set of all $S \in \mathcal{F}(G)$ with $|S| \leq \mathsf{D}(G)$. But with the algorithm presented below, we can avoid most of the redundant checks and therefore speed up the computation dramatically.

Algorithm 1 Recursive Atom Search: $A \leftarrow \mathrm{RAS}(A, S, \Sigma, B)$

```
for all g ∈ B do
  S' ← Sg
  if g ≤ −σ(S') then
    A ← A ∪ {S'(−σ(S'))}
  end if
  Σ' ← Σ
  B' ← ∅
  for all g' ∈ B do
    if g + g' ∈ Σ then
      Σ' ← Σ' ∪ {g'}
    else
      B' ← B' ∪ {g'}
    end if
  end for
  if #B' > 0 then
    A ← RAS(A, S', Σ', B')
  end if
end for
return A
```

Algorithm 2 Atoms Computation Algorithm 1: $\mathcal{A}(G) \leftarrow \mathrm{ACA1}(G)$

```
A ← {0}
for all g ∈ G \ {0} do
  if g ≤ −g then
    A ← A ∪ {g(−g)}
  end if
  Σ ← {0, g}
  B ← G \ {0, −g}
  S ← g
  if #B > 0 then
    A ← RAS(A, S, Σ, B)
  end if
end for
return A
```

Since modular arithmetic on vectors with multiple coordinates is quite inefficient, it is necessary for a fast execution of the RAS, Algorithm 2.3.2, to pre-compute the sums $g + g'$. This can be done once in the ACA1, Algorithm 2.3.2, before the main loop. For additional details on speeding up this type of algorithms by special alignment of the pre-computed data and on the parallelization aspects, the reader is referred to [**15**, Section 3].

2.3.3. The set of atoms of a T-block monoid.

LEMMA 2.3.1. *Let G be a finite additive abelian group, T a reduced atomic monoid, $\iota : T \to G$ a homomorphism, and $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$ the T-block monoid over G defined* *by ι. Furthermore, let each class in G contain some $p \in P$, and let $\bar{\iota} : \mathsf{Z}(T) \to \mathcal{F}(G)$ be the homomorphism generated by the extension of ι onto $\mathsf{Z}(T)$ such that, for a factorization $z = a_1 \cdot \ldots \cdot a_n \in \mathsf{Z}(T)$ with $a_i \in \mathcal{A}(T)$ for $i \in [1, n]$, we have $\bar{\iota}(z) = \iota(a_1) \cdot \ldots \cdot \iota(a_n)$. Then we have*

$$(2.3.2) \quad \mathcal{A}(\mathcal{B}(G, T, \iota)) = \{S\pi(z) \mid S \in \mathcal{F}(G),\, z \in \mathsf{Z}(T),\, S\bar{\iota}(z) \in \mathcal{A}(G), \nexists n \geq 2 : \exists S_i \in \mathcal{F}(G),\, z_i \in \mathsf{Z}(T) \text{ with } S_i\bar{\iota}(z_i) \in \mathcal{A}(G) \text{ for } i \in [1, n] : S_1\pi(z_1) \cdot \ldots \cdot S_n\pi(z_n) = S\pi(z)\}$$

Proof. Clearly, every atom $a \in \mathcal{A}(\mathcal{B}(G, T, \iota))$ is of the form $a = S\pi(z)$ with $S \in \mathcal{F}(G)$, $z \in \mathsf{Z}(T)$, and $S\bar{\iota}(z) \in \mathcal{A}(G)$. Now suppose we have $n \in [2, \mathsf{D}(G)]$, $S_i \in \mathcal{F}(G)$, $z_i \in \mathsf{Z}(T)$, $S_i\bar{\iota}(z_i) \in \mathcal{A}(G)$ for $i \in [1, n]$ and $S\pi(z) = S_1\pi(z_1) \cdot \ldots \cdot S_n\pi(z_n)$. Obviously then, $a \notin \mathcal{A}(\mathcal{B}(G, T, \iota))$. Now the other inclusion is obvious. $\square$

In general, it is very hard to calculate $\mathcal{A}(\mathcal{B}(G, T, \iota))$ explicitly by the characterization in (2.3.2). But if we restrict ourselves to a finite group G and a finitely generated reduced monoid T such that $\mathcal{A}(G)$, $\mathcal{A}(T)$, and $\iota(a)$ for $a \in \mathcal{A}(T)$ are all known explicitly, we can formulate Algorithm 2.3.3 for the computation of the set of atoms of a T-block monoid.

2.3.4. Computing arithmetical invariants of a T-block monoid.

Throughout this section, we silently use the isomorphism defined in (2.3.1). Thus we only have to work with submonoids

$$S \subset \mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_r\mathbb{Z} \quad \text{with } m,\, r \in \mathbb{N}_0 \text{ and } n_1, \ldots, n_r \in \mathbb{N}$$

such that $S \cong \varphi(\mathcal{B}(G, T, \iota))$ (identify!), where G is an additively written finite abelian group, T is a product of finitely many reduced finitely primary monoids of rank 1, $\iota : G \to T$ is a homomorphism, and φ is the isomorphism defined in (2.3.1). If T is not the product of only finitely many reduced finitely primary monoids of rank 1, then T would not be finitely generated. Now we know $\mathcal{A}(S)$ explicitly, since, obviously, $\mathcal{A}(S) = \varphi(\mathcal{A}(\mathcal{B}(G, T, \iota)))$ and $\mathcal{A}(\mathcal{B}(G, T, \iota))$ can be computed explicitly by the ACA2, see Algorithm 2.3.3.

For the computation of the tame degree, we use Definition 2.1.13 and Theorem 2.1.14.2; for additional reference on this computation, see [**6**, Section 4].

Now we are ready to describe the computation step by step.

2.3.4.1. *Finding the elements of $\mathcal{A}(\sim_S)$.*
The first step is finding the elements of $\mathcal{A}(\sim_S)$ explicitly. Unfortunately, this is a very hard task. Probably, the most efficient way is the following one as described in [**7**, Sections 1 and 2].

1. Since we know $\mathcal{A}(S)$ explicitly, we can write the atoms of S in their coordinates as vectors:

$$\mathcal{A}(S) = \{(a_1^{(1)}, \ldots, a_m^{(1)}, a_{m+1}^{(1)} \mod n_1, \ldots, a_{m+r}^{(1)} \mod n_r), \ldots\}.$$

Algorithm 3 Atoms Computation Alg. 2: $\mathcal{A}(\mathcal{B}(G,T,\iota)) \leftarrow \mathrm{AA2}(G,T,\mathcal{A}(G),\mathcal{A}(T),\iota)$

```
A ← ∅
D ← 0
for all S ∈ A(G) do
  if |S| > D then
    D ← |S|
  end if
  A ← A ∪ {(S, 1)}
end for
F_0 ← ∅
for all a ∈ A(T) do
  for all (S, 1) ∈ A do
    if ι(a) | S then
      F_0 ← F_0 ∪ {(ι(a)^{-1}S, a)}
    end if
  end for
end for
E ← ∅
n ← 1
while n < D and F_{n-1} ≠ ∅ do
  E ← E ∪ F_{n-1}
  E ← E F_0
  F_n ← ∅
  for all a ∈ A(T) do
    for all (S, b) ∈ A do
      if ι(a) | S then
        F_n ← F_n ∪ {(ι(a)^{-1}S, ab)
      end if
    end for
  end for
  n ← n + 1
end while
return A ∪ F_0 ∪ ... ∪ F_{n-1}
```

2. By [**7**, Section 2], finding the elements of $\mathcal{A}(\sim_S)$ is equivalent to determining the minimal positive solutions of the following system of linear diophantine equations:

(2.3.3)
$$\begin{array}{ccccccccccccc}
x_1a_1^{(1)} & + & \dots & + & x_ka_1^{(k)} & - & y_1a_1^{(1)} & - & \dots & - & y_ka_1^{(k)} & = 0 & \\
 & & \vdots & & & & & & \vdots & & & & \\
x_1a_{m+r}^{(1)} & + & \dots & + & x_ka_{m+r}^{(k)} & - & y_1a_{m+r}^{(1)} & - & \dots & - & y_ka_{m+r}^{(k)} & \equiv 0 & \mod n_r
\end{array}$$

We write a solution $(x_1,\dots,x_k,y_1,\dots,y_k)$ as $((x_1,\dots,x_k),(y_1,\dots,y_k))$.

3. Again, by [**7**, Section 2] and by [**23**, Section 2], finding the set of minimal positive solutions is equivalent to finding the set of minimal positive solutions for the

following enlarged system and then projecting back by the map and removing the zero element (if appearing after the projection) from the set of solutions:

$$\begin{array}{ccccccccccccc} x_1 a_1^{(1)} & + & \dots & - & y_1 a_1^{(1)} & - & \dots & & & & & = 0 \\ & & \vdots & & & & \vdots & & & & & \\ x_1 a_{m+r}^{(1)} & + & \dots & - & y_1 a_{m+r}^{(1)} & - & \dots & + & x_{k+r} n_r & - & y_{k+r} n_r & = 0 \end{array} \tag{2.3.4}$$

$$\Phi : \begin{cases} \mathbb{N}_0^{k+r} \times \mathbb{N}_0^{k+r} & \to & \mathbb{N}_0^{k} \times \mathbb{N}_0^{k} \\ ((x_1, \dots, x_{k+r}), (y_1, \dots, y_{k+r})) & \mapsto & ((x_1, \dots, x_k), (y_1, \dots, y_k)). \end{cases}$$

One of the most efficient algorithms for finding these solutions is due to Contejean and Devie; see [**8**]. Nevertheless, this might take a very long time since the problem of determining the set of all minimal non-negative solutions of a system of linear diophantine equations is well known to be NP-complete.

2.3.4.2. *Removing unnecessary elements.*
Clearly, elements of the form $((1, 0, \dots, 0),\ (1, 0, \dots, 0))$ are minimal solutions. But as elements of $\mathcal{A}(\sim_S)$, these elements do not carry any information about the arithmetic of S. Therefore we may simply drop them. Since, for any two factorizations, $(x, y) \in \mathsf{Z}(S)$ is equivalent to $(y, x) \in \mathsf{Z}(S)$, we may also reduce the number of pairs by a factor of two. This smaller set will be denoted by $\mathcal{A}(\sim_S)^* = \{((x_1, \dots, x_k), (y_1, \dots, y_k)), \dots\}$.

2.3.4.3. *Computing the elasticity.*
By our finiteness assumptions on T, i.e. T is finitely generated, we know this set is finite. Thus we can simply compute the elasticity using Proposition 2.1.9.2 as follows:

$$\rho(S) = \max \left\{ \frac{x_1 + \dots + x_k}{y_1 + \dots + y_k}, \frac{y_1 + \dots + y_k}{x_1 + \dots + x_k} \,\middle|\, ((x_1, \dots, x_k), (y_1, \dots, y_k)) \in \mathcal{A}(\sim_S)^* \right\}.$$

2.3.4.4. *Computing the catenary degree.*
By Equation 2.1.1, we have to consider only elements $a \in S$ such that their factorizations appear as part of an element of $\mathcal{A}(\sim_S)$ and such that their sets of factorizations consist of more than one $\mathcal{R}$-equivalence class. Then we get the catenary degree by taking the maximum over $\mu(a)$ for all those $a \in S$.

2.3.4.5. *Computing the tame degree.*
After having computed $\mathsf{Z}(a)$ for all $a \in S$ such that $\mathcal{A}_a(\sim_S) \neq \emptyset$, we can apply Theorem 2.1.14.1 for every $u \in \mathcal{A}(S)$. Since there are only finitely many, we get the tame degree as the maximum of these values.

2.3.4.6. *Computing the monotone catenary degree.* For computing the monotone catenary degree, we compute the equal catenary degree $\mathsf{c}_{\mathrm{eq}}(S)$ and the adjacent catenary degree $\mathsf{c}_{\mathrm{ad}}(S)$. We start with the adjacent catenary degree and proceed like in 2.3.4.1. We use the fact that $\sim_{S,\mathrm{mon}} = \{(x, y) \in \sim_S \mid |x| \le |y|\}$ and again [**7**, Section 2]. Now finding the elements of $\mathcal{A}(\sim_{S,\mathrm{mon}})$ is equivalent to determining the minimal positive solutions of a system of linear diophantine equations.

Before we construct this finite system of linear diophantine equations explicitly, we formulate a short lemma.

LEMMA 2.3.2. *Let H be a reduced finitely generated atomic monoid. Then $\sim_{H,\mathrm{mon}}$ is finitely generated.*

Algorithm 4 Recursive $\mathcal{R}$-Class Finder: $\mathcal{R} \leftarrow \mathrm{RCF}(\mathcal{R}, \mathsf{Z} = \{z_1, \ldots, z_n\})$

```
r ← {z_1}
Z ← Z \ {z_1}
n ← n − 1
Z = {z_1, ..., z_n} {renumber}
i ← 1
while i < n do
  for i = 1 to n do
    if gcd(z_i, x) ≠ 1 for some x ∈ r then
      r ← r ∪ {z_i}
      Z ← Z \ {z_i}
      n ← n − 1
      Z = {z_1, ..., z_n} {renumber}
      break
    end if
  end for
end while
R ∪ {r}
if Z ≠ ∅ then
  R ← RCF(R, Z)
end if
return R
```

Algorithm 5 Catenary degree Computation Algorithm: $\mathsf{c}(S) \leftarrow \mathrm{CCA}(\mathcal{A}(S), \mathcal{A}(\sim_S)^*)$

```
A ← ∅
for all (x, y) ∈ A(~_S)^* do
  A ← A ∪ {π(x)}
end for
c ← 0
for all a ∈ A do
  R_a ← RCF(Z(a))
  if #R_a > 1 then
    μ ← min{|x| | R_a}
    if c < μ then
      c ← μ
    end if
  end if
end for
return c
```

Proof. Let H be a reduced finitely generated atomic monoid. Since $\sim_H \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ is then a saturated submonoid of a finitely generated monoid, $\sim_H$ is finitely generated by [**14**, Proposition 2.7.5]. Now assume $\sim_H$ has $n \in \mathbb{N}$ generators. Then the atoms of $\sim_H$ can be described as the minimal solutions of a system of finitely many, say k, linear diophantine equations in $2n$ variables as in step 2.3.4.1 above. Then the atoms of $\sim_{H,\mathrm{mon}}$ can be described as the minimal solutions of a system of $k+1$ linear diophantine equations in $2n+1$ variables—see below for the explicit description of this system of linear diophantine equations. Thus $\sim_{H,\mathrm{mon}}$ is finitely generated. $\square$

The system is (2.3.3), with one additional variable z and one equation, namely,

$$x_1 + \ldots + x_k - y_1 - \ldots - y_k + z = 0.$$

The coefficients at z are zero in all other equations. Now we have two possibilities.

- Either we proceed by the same steps as in 2.3.4.1 and solve this directly
- or we use the incremental version of the algorithm of Devie and Contejoud (see [**8**, Section 9]) and the set $\mathcal{A}(\sim_S)$, which we already computed in 2.3.4.1.

Next we can reduce the set of relations, which we must consider, as in 2.3.4.2. By Theorem 2.2.16, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\mathrm{mon}}) \neq \emptyset$. Then we get the adjacent catenary degree by taking the maximum over $\mu_{\mathrm{ad}}(a)$ for all those a. For the computation of the equal catenary degree, we must know the elements of $\mathcal{A}(\sim_{S,\mathrm{eq}})$. But these are already known, since $\mathcal{A}(\sim_{S,\mathrm{eq}}) \subset \mathcal{A}(\sim_{S,\mathrm{mon}})$. Here we can again reduce the set of relations which we must consider, as in 2.3.4.2. By Theorem 2.2.12, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\mathrm{eq}}) \neq \emptyset$ and $|\mathcal{R}_{a,k}| > 1$ for some $k \in \mathsf{L}(a)$. Now this can be done by applying Algorithm 4 on $\mathsf{Z}_k(a)$ instead of $\mathsf{Z}(a)$. Then we get the equal catenary degree by taking the maximum over $\mu_{\mathrm{eq}}(a)$ for all those a.

Now we find the monotone catenary degree by $\mathsf{c}_{\mathrm{mon}}(S) = \max\{\mathsf{c}_{\mathrm{ad}}(S), \mathsf{c}_{\mathrm{eq}}(S)\}$.

2.3.4.7. *Reducing the computation time for the catenary degree.*
If we are only interested in the computation of the catenary degree, we can speed up the very time consuming computations in step 2.3.4.1 in the following way. In favor of Equation 2.1.1, we may restrict our search for minimal solutions of the system of linear diophantine equations (2.3.4) to solutions $(x_1, \ldots, x_{k+r}, y_1, \ldots, y_{k+r})$ such that $\sum_{i=1}^k x_i \le \mathsf{c}(S)$ and $\sum_{i=1}^k y_i \le \mathsf{c}(S)$. Of course, we do not know $\mathsf{c}(S)$ a priori, but we may replace it with any upper bound—the better the bound, the faster the computation. In our special situation of T-block monoids, we can find a reasonably good bound by [**14**, Theorem 3.6.4.1] and by [**14**, Proposition 3.6.6]. Formulated in our terminology, these results read as follows.

THEOREM 2.3.3. *Let G be an additively written abelian group, T a reduced finitely generated monoid, $\iota : T \to G$ a homomorphism, and $\mathcal{B}(G,T,\iota) \subset \mathcal{F}(G) \times T$ the T-block monoid over G defined by ι. Then*

1. $\rho(\mathcal{B}(G,T,\iota), \mathcal{F}(G) \times T) \le \rho(T)$.
2. $\mathsf{c}(\mathcal{B}(G,T,\iota)) \le \rho(T)\mathsf{D}(G)\max\{\mathsf{c}(T), \mathsf{D}(G)\}$.

Now we set $C = \rho(T)\mathsf{D}(G)\max\{\mathsf{c}(T), \mathsf{D}(G)\}$ for the upper bound. Though this does not speed up the search for minimal solutions itself that much, it is a very efficient (additional) termination criterion in our variant of the algorithm due to Contejean and Devie; for reference on the originally proposed algorithm, see [**8**].
Unfortunately, this method has one drawback for the computation of the elasticity and the tame degree. As we no longer compute all minimal solutions to our system of linear diophantine equations, we no longer compute all elements in $\mathcal{A}(\sim_S)$, and therefore we

cannot compute more than a lower bound for the elasticity in step 2.3.4.3 and for the tame degree in step 2.3.4.5.

2.3.4.8. *Computing the elasticity from an appropriate subset of* $\mathcal{A}(\sim_S)$.
In [**9**], Domenjoud proposed an algorithm for computing the set of minimal solutions of a system of linear diophantine equations, which computes the set of minimal solutions with minimal support in a first step. All other minimal solutions can then be found by "appropriate" linear combinations of them using non-negative rational coefficients. With this interesting fact in mind, we consider the following lemma.

Definition 2.3.4. Let H be a reduced atomic monoid. For $x \in \mathsf{Z}(H)$, we set

$$\operatorname{supp}(x) = \{u \in \mathcal{A}(H) \mid u \mid x\}.$$

Lemma 2.3.5. *Let H be a finitely generated atomic monoid. Then*

$$\rho(H) = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x,y) \in \mathcal{A}'(\sim_H) \right\},$$

where $\mathcal{A}'(\sim_H) = \{(x,y) \in \mathcal{A}(\sim_H) \mid \operatorname{supp}(x) \cup \operatorname{supp}(y)$ *is minimal*$\}$.

Proof. Let $(x,y) \in \mathcal{A}(\sim_H)$. Then there are $n \in \mathbb{N}$, $(x_i, y_i) \in \mathcal{A}'(\sim_H)$, $q_i \in \mathbb{Q}$ with $0 \le q_i < 1$ for $i \in [1,n]$ such that

$$(x,y) = \prod_{i=1}^{n} (x_i, y_i)^{q_i}.$$

Such a decomposition exists, since the equivalent one exists for the set of solutions of the associated system of linear diophantine equations, see [**9**, Theorem 3]. When we pass to the lengths, we find $|x| = \sum_{i=1}^{n} q_i |x_i|$ and $|y| = \sum_{i=1}^{n} q_i |y_i|$. This yields

$$\frac{|x|}{|y|} \cdot |y| = |x| = \sum_{i=1}^{n} q_i |x_i| = \sum_{i=1}^{n} q_i \frac{|x_i|}{|y_i|} |y_i| \le \max_{i=1}^{n} \frac{|x_i|}{|y_i|} \sum_{i=1}^{n} q_i |y_i| = \max_{i=1}^{n} \frac{|x_i|}{|y_i|} \cdot |y| .$$

Thus we find

$$\frac{|x|}{|y|} \le \max_{i=1}^{n} \frac{|x_i|}{|y_i|}.$$

Since $\mathcal{A}'(\sim_H) \subset \mathcal{A}(\sim_H)$, the assertion now follows by Proposition 2.1.9.2. □

Thus we can restrict ourselves to the minimal solutions with minimal support for computing the elasticity.
As far as computational performance is concerned, the most interesting point of this approach is that there are straightforward optimizations of Domenjoud's algorithm for symmetric systems of linear diophantine equations like the one in (2.3.4).

2.3.5. Explicit examples $\mathbb{F}_3[X^2, X^3]$, $\mathbb{F}_2[X^2, X^3]$, and $\mathbb{F}_2[X^2, X^5]$.

Let $p \in \{2,3\}$. Let $R = \mathbb{F}_p[X^2, X^3]$. Then R is a one-dimensional noetherian domain with integral closure $\widehat{R} = \mathbb{F}_p[X]$ and conductor $\mathfrak{f} = (R : \widehat{R}) = X^2 \widehat{R}$, where $X \in \widehat{R}$ is a prime element. Thus R is an order in the Dedekind domain $\widehat{R}$, and $X\widehat{R}$ is the only maximal ideal of $\widehat{R}$ containing $\mathfrak{f}$. Furthermore, $\widehat{R}^\times = R^\times = \mathbb{F}_p^\times$. By the computations in [**14**, Special case 3.2 in Example 3.7.3], we have $G = \operatorname{Pic}(R) \cong \mathbb{F}_p$.

2.3.5.1. $\mathbb{F}_3[X^2, X^3]$.

Now let $p = 3$. Since $\#G = 3$, we write $G = \{\mathbf{0}, \mathbf{e}, \mathbf{e}'\}$. Clearly—or by applying the ACA1, see Algorithm 2.3.2—we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}\mathbf{e}', \mathbf{e}^3, \mathbf{e}'^3\}$. Now we apply [**14**, Theorem 3.7.1] and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2,3\},\, g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2,3\}$ and $g \in G$.

Now we apply the ACA2 (see Algorithm 2.3.3).

1. $F_0 = \{(\mathbf{0}, 1), (\mathbf{e}\mathbf{e}', 1), (\mathbf{e}^3, 1), (\mathbf{e}'^3, 1)\}$
2. $A = \{(X^n g, g) \mid n \in \{2,3\},\, g \in G\}$
3. $(S, 1) \in F_0$, $S = \mathbf{0}$
   (a) $(X^k\mathbf{0}, \mathbf{0}) \in B = \{(X^n, \mathbf{0}) \mid n \in \{2,3\}\}$, $k \in \{2,3\}$
   (b) $F_1 = \{(1, X^n\mathbf{0}) \mid n \in \{2,3\}\}$
4. $(S, 1) \in F_0$, $S = \mathbf{e}\mathbf{e}'$
   (a) $(X^k g, g) \in B = \{(X^n, h) \mid n \in \{2,3\},\, h \in \{\mathbf{e}, \mathbf{e}'\}\}$, $k \in \{2,3\}$, $g \in \{\mathbf{e}, \mathbf{e}'\}$
   (b) $F_1 = F_1 \cup \{(g', X^k g) \mid k \in \{2,3\},\, \{g, g'\} = \{\mathbf{e}, \mathbf{e}'\}\}$
5. $(S, 1) \in F_0$, $S = g^3$, $g \in \{\mathbf{e}, \mathbf{e}'\}$
   (a) $(X^k g, g) \in B = \{(X^n g, g) \mid n \in \{2,3\},\, k \in \{2,3\}$
   (b) $F_1 = F_1 \cup \{(g^2, X^n g) \mid n \in \{2,3\}$
   $F_1 = \{(1, X^n\mathbf{0}) \mid n \in \{2,3\}\} \cup \{(g, X^n g') \mid n \in \{2,3\},\, \{g, g'\} = \{\mathbf{e}, \mathbf{e}'\}\} \cup \{(g^2, X^n g) \mid n \in \{2,3\},\, g \in \{\mathbf{e}, \mathbf{e}'\}\}$
6. $n = 2$
7. $n \le \mathsf{D}(G) = 3$ and $F_1 \ne \emptyset$: *true*
   (a) $E = F_1 F_1$
   (b) $F_2 = \emptyset$
   (c) $(S, t) \in F_1$, $S = g$, $g \in \{\mathbf{e}, \mathbf{e}'\}$
      (i) $(X^n g, g) \in B = \{(X^n g, g) \mid n \in \{2,3\}\}$, $g \in \{\mathbf{e}, \mathbf{e}'\}$
      (ii) $(S', t') = (1, X^k(g + g')) = (1, X^k\mathbf{0})$, $k \in \{4,5,6\}$, $\{g, g'\} = \{\mathbf{e}, \mathbf{e}'\}$
      (iii) $(S', t') = (1, X^{k'}\mathbf{0})(1, X^{k''}\mathbf{0}) \in E = F_1 F_1$, $k' + k'' = k$, $k', k'' \in \{2,3\}$
   (d) $(S, t) \in F_1$, $S = g^2$, $g \in \{\mathbf{e}, \mathbf{e}'\}$
      (i) $(X^k g, g) \in B = \{(X^n g, g) \mid n \in \{2,3\},\, k \in \{2,3\},\, g \in \{\mathbf{e}, \mathbf{e}'\}$
      (ii) $(S', t') = (g, X^k(g + g)) = (g, X^k g')$, $k \in \{4,5,6\}$, $\{g, g'\} = \{\mathbf{e}, \mathbf{e}'\}$
      (iii) $(S', t') = (1, X^{k'}\mathbf{0})(g, X^{k''} g') \in F_1 F_1$
   (e) $n = n + 1 = 3$
   $F_2 = \emptyset$
8. $n \le 3$ and $F_2 \ne \emptyset$: *false*
9. $\mathcal{A}(\mathcal{B}(G, T, \iota)) = \bigcup_{i=0}^{n-1} F_i = F_0 \cup F_1 \cup F_2 = F_0 \cup F_1$

Finally, we find

$$\begin{aligned}\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{&(\mathbf{0}, 1), (\mathbf{e}\mathbf{e}', 1), (\mathbf{e}^3, 1), (\mathbf{e}'^3, 1), (1, X^2\mathbf{0}), (1, X^3\mathbf{0}), (\mathbf{e}, X^2\mathbf{e}'), (\mathbf{e}, X^3\mathbf{e}'),\\ &(\mathbf{e}', X^2\mathbf{e}), (\mathbf{e}', X^3\mathbf{e}), (\mathbf{e}^2, X^2\mathbf{e}), (\mathbf{e}^2, X^3\mathbf{e}), (\mathbf{e}'^2, X^2\mathbf{e}'), (\mathbf{e}'^2, X^3\mathbf{e}')\}.\end{aligned}$$

Using the construction from the beginning of subsection 2.3.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^4 \times \mathbb{Z}/3\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\begin{aligned}\mathcal{A}(S) = \{&(1,0,0,0,\bar{0}),(0,1,1,0,\bar{0}),(0,3,0,0,\bar{0}),(0,0,3,0,\bar{0}),(0,0,0,2,\bar{0}),\\ &(0,0,0,3,\bar{0}),(0,1,0,2,\bar{2}),(0,1,0,3,\bar{2}),(0,0,1,2,\bar{1}),(0,0,1,3,\bar{1}),\\ &(0,2,0,2,\bar{1}),(0,2,0,3,\bar{1}),(0,0,2,2,\bar{2}),(0,0,2,3,\bar{2})\}\,.\end{aligned}$$

Since the atom $(1,0,0,0,\bar{0})$ is prime, we can restrict on a monoid $\bar{S} \subset \mathbb{N}_0^3 \times Z/3\mathbb{Z}$ with the following set of atoms

$$\begin{aligned}\mathcal{A}(\bar{S}) = \{&(1,1,0,\bar{0}),(3,0,0,\bar{0}),(0,3,0,\bar{0}),(0,0,2,\bar{0}),(0,0,3,\bar{0}),(1,0,2,\bar{2}),\\ &(1,0,3,\bar{2}),(0,1,2,\bar{1}),(0,1,3,\bar{1}),(2,0,2,\bar{1}),(2,0,3,\bar{1}),(0,2,2,\bar{2}),(0,2,3,\bar{2})\}\,.\end{aligned}$$

Now we can find everything by using the algorithms presented at the end of subsection 2.3.4.

Even in the modified version of the algorithm in step 2.3.4.1—here the bound is 13.5—we find about 7,500 minimal representations to consider after the reduction in step 2.3.4.2.

From those, we get $\mathsf{c}(\mathbb{F}_3[X^2,X^3]) = 3$ in step 2.3.4.4. Since we did not compute all minimal solutions, we find $\mathsf{t}(\mathbb{F}_3[X^2,X^3]) \geq 4$ in step 2.3.4.5.
By using the alternative approach from subsubsection 2.3.4.8, we find $\rho(\mathbb{F}_3[X^2,X^3]) = 2$.

2.3.5.2. $\mathbb{F}_2[X^2,X^3]$.
Let $p = 2$. Then $\#G = 2$, write $G = \{\mathbf{0},\mathbf{e}\}$. Obviously—or by applying the ACA1, see Algorithm 2.3.2—we have $\mathcal{A}(G) = \{\mathbf{0},\mathbf{e}^2\}$. Now we apply [**14**, Theorem 3.7.1] as in the case $p = 3$ and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G,T,\iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2,3\},\, g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2,3\}$ and $g \in G$.
Now we apply the ACA2, see Algorithm 2.3.3, as before and find

$$\mathcal{A}(\mathcal{B}(G,T,\iota)) = \{(\mathbf{0},1),(\mathbf{e}^2,1),(1,X^2\mathbf{0}),(1,X^3\mathbf{0}),(\mathbf{e},X^2\mathbf{e}),(\mathbf{e},X^3\mathbf{e})\}\,.$$

Using the construction from the beginning of subsection 2.3.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G,T,\iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1,0,0,\bar{0}),(0,2,0,\bar{0}),(0,0,2,\bar{0}),(0,0,3,\bar{0}),(0,1,2,\bar{1}),(0,1,3,\bar{1})\}\,.$$

Since the atom $(1,0,0,\bar{0})$ is prime, we can use the same arguments as in Lemma 1.2.16.4 and restrict on a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with the following set of atoms

$$\mathcal{A}(\bar{S}) = \{(2,0,\bar{0}),(0,2,\bar{1}),(0,3,\bar{1}),(1,2,\bar{1}),(1,3,\bar{1})\}\,.$$

By Theorem 2.3.3, we find

$$\mathsf{c}(\mathbb{F}_2[X^2,X^3]) \leq \rho(T)\mathsf{D}(G)\max\{\mathsf{c}(T),\mathsf{D}(G)\} = 9\,.$$

Since in this case step 2.3.4.1 can be performed easily without this bound, we compute all atoms. Now, we find the following list of atoms after step 2.3.4.2:

$$((0,0,1,3,0),(0,0,0,0,3)),((0,1,0,0,1),(0,0,1,1,0)),((0,2,0,1,0),(0,0,1,0,1)),$$
$$((0,3,0,0,0),(0,0,2,0,0)),((1,0,2,0,0),(0,0,0,0,2)),((1,2,1,1,0),(0,0,0,0,3)),$$
$$((1,3,0,0,0),(0,0,0,0,2)),((1,0,0,0,4),(0,0,0,6,0)),((1,0,1,0,1),(0,0,0,3,0)),$$
$$((2,0,3,0,0),(0,0,0,3,1)),((3,0,4,0,0),(0,0,0,6,0)),((1,0,2,0,0),(0,1,0,2,0)),$$
$$((1,0,3,0,0),(0,2,0,1,1)),((1,1,1,0,0),(0,0,0,1,1)),((1,4,0,0,0),(0,0,1,1,1)),$$
$$((1,1,0,0,2),(0,0,0,4,0)),((2,1,2,0,0),(0,0,0,4,0)),((1,2,0,0,0),(0,0,0,2,0)).$$

Given this list, we immediately find $\rho(\mathbb{F}_2[X^2,X^3])=2$ in step 2.3.4.3.
Now we proceed with step 2.3.4.4. First we compute the elements involved in the atoms and find

$$(3,9,1),\ (1,5,1),\ (1,6,1),\ (0,6,0),\ (2,6,0),\ (6,12,0),\ (3,6,1),$$
$$(4,9,0),\ (2,9,0),\ (2,5,0),\ (2,8,0),\ (4,8,0),\ (2,4,0).$$

Now we compute all factorizations for each of these elements and their decompositions into $\mathcal{R}$-classes:

(3, 9, 1)	[(1, 3, 0, 0, 1), (1, 2, 1, 1, 0), (1, 0, 2, 0, 1), (0, 1, 0, 2, 1), (0, 0, 1, 3, 0), (0, 0, 0, 0, 3)]	1 $\mathcal{R}$-class
(1, 5, 1)	[(0, 1, 0, 0, 1)] [(0, 0, 1, 1, 0)]	2 $\mathcal{R}$-classes
(1, 6, 1)	[(0, 2, 0, 1, 0)] [(0, 0, 1, 0, 1)]	2 $\mathcal{R}$-classes
(0, 6, 0)	[(0, 3, 0, 0, 0)] [(0, 0, 2, 0, 0)]	2 $\mathcal{R}$-classes
(2, 6, 0)	[(1, 3, 0, 0, 0), (1, 0, 2, 0, 0), (0, 1, 0, 2, 0)] [(0, 0, 0, 0, 2)]	2 $\mathcal{R}$-classes
(6, 12, 0)	[(3, 6, 0, 0, 0), (3, 3, 2, 0, 0), (3, 0, 4, 0, 0), (2, 4, 0, 2, 0), (2, 3, 0, 0, 2), (2, 2, 1, 1, 1), (2, 1, 2, 2, 0), (2, 0, 2, 0, 2), (1, 2, 0, 4, 0), (1, 1, 0, 2, 2), (1, 0, 1, 3, 1), (1, 0, 0, 0, 4), (0, 0, 0, 6, 0)]	1 $\mathcal{R}$-class
(3, 6, 1)	[(1, 2, 0, 1, 0), (1, 0, 1, 0, 1), (0, 0, 0, 3, 0)]	1 $\mathcal{R}$-class
(4, 9, 0)	[(2, 3, 1, 0, 0), (2, 0, 3, 0, 0), (1, 2, 0, 1, 1), (1, 1, 1, 2, 0), (1, 0, 1, 0, 2), (0, 0, 0, 3, 1)]	1 $\mathcal{R}$-class
(2, 9, 0)	[(1, 3, 1, 0, 0), (1, 0, 3, 0, 0), (0, 2, 0, 1, 1), (0, 1, 1, 2, 0), (0, 0, 1, 0, 2)]	1 $\mathcal{R}$-class
(2, 5, 0)	[(1, 1, 1, 0, 0)] [(0, 0, 0, 1, 1)]	2 $\mathcal{R}$-classes
(2, 8, 0)	[(1, 4, 0, 0, 0), (1, 1, 2, 0, 0), (0, 2, 0, 2, 0), (0, 1, 0, 0, 2), (0, 0, 1, 1, 1)]	1 $\mathcal{R}$-class
(4, 8, 0)	[(2, 4, 0, 0, 0), (2, 1, 2, 0, 0), (1, 2, 0, 2, 0), (1, 1, 0, 0, 2), (1, 0, 1, 1, 1), (0, 0, 0, 4, 0)]	1 $\mathcal{R}$-class
(2, 4, 0)	[(1, 2, 0, 0, 0)] [(0, 0, 0, 2, 0)]	2 $\mathcal{R}$-classes

From this one deduces $\mathsf{c}(\mathbb{F}_2[X^2,X^3])=3$ and $\mathsf{D}(\mathbb{F}_2)+1+\mathsf{t}(S)=6\geq\mathsf{t}(\mathbb{F}_2[X^2,X^3])\geq\mathsf{t}(S)=3$.

Next, we compute the monotone catenary degree. For this, we proceed as in step 2.3.4.6 and start with the adjacent catenary degree. We find the following list of atoms of the monoid of monotone relations:

$$((0,0,0,0,3),(0,0,1,3,0)),\ ((0,0,0,0,2),(0,1,0,2,0)),\ ((0,0,0,0,2),(1,0,2,0,0)),$$
$$((0,0,0,0,3),(1,2,1,1,0)),\ ((0,0,0,0,2),(1,3,0,0,0)),\ ((0,0,0,3,1),(2,0,3,0,0)),$$

$$((0,0,0,1,1),(1,1,1,0,0)),\ ((0,0,0,2,0),(1,2,0,0,0)),\ ((0,0,1,0,1),(0,2,0,1,0)),$$
$$((0,0,2,0,0),(0,3,0,0,0)),\ ((0,0,1,1,1),(1,4,0,0,0)),\ ((0,1,0,0,1),(0,0,1,1,0)),$$
$$((1,0,0,0,4),(0,0,0,6,0)),\ ((1,0,1,0,1),(0,0,0,3,0)),\ ((1,1,0,0,2),(0,0,0,4,0)),$$
$$((1,0,2,0,0),(0,1,0,2,0)),\ ((1,0,3,0,0),(0,2,0,1,1)),\ ((1,0,4,0,0),(0,3,0,0,2)).$$

Next, we compute the elements involved in the atoms and find

$$(3,9,1),\ (2,6,0),\ (4,9,0),\ (2,5,0),\ (2,4,0),\ (1,6,1),\ (0,6,0),$$
$$(2,8,0),\ (1,5,1),\ (6,12,0),\ (3,6,1),\ (4,8,0),\ (2,9,0),\ (2,12,0).$$

The factorizations of these elements sorted by their lengths are

(3,9,1)	3	(0, 0, 0, 0, 3)
	4	(1, 0, 2, 0, 1), (0, 1, 0, 2, 1), (0, 0, 1, 3, 0)
	5	(1, 3, 0, 0, 1), (1, 2, 1, 1, 0)
(2,6,0)	2	(0, 0, 0, 0, 2)
	3	(1, 0, 2, 0, 0), (0, 1, 0, 2, 0)
	4	(1, 3, 0, 0, 0)
(4,9,0)	4	(1, 0, 1, 0, 2), (0, 0, 0, 3, 1)
	5	(2, 0, 3, 0, 0), (1, 2, 0, 1, 1), (1, 1, 1, 2, 0)
	6	(2, 3, 1, 0, 0)
(2,5,0)	3	(0, 0, 0, 1, 1)
	3	(1, 1, 1, 0, 0)
(2,4,0)	2	(0, 0, 0, 2, 0)
	3	(1, 2, 0, 0, 0)
(1,6,1)	2	(0, 0, 1, 0, 1)
	3	(0, 2, 0, 1, 0)
(0,6,0)	2	(0, 0, 2, 0, 0)
	3	(0, 3, 0, 0, 0)
(2,8,0)	3	(0, 1, 0, 0, 2), (0, 0, 1, 1, 1)
	4	(1, 1, 2, 0, 0), (0, 2, 0, 2, 0)
	5	(1, 4, 0, 0, 0)
(1,5,1)	2	(0, 1, 0, 0, 1), (0, 0, 1, 1, 0)
(6,12,0)	5	(1, 0, 0, 0, 4)
	6	(2, 0, 2, 0, 2), (1, 1, 0, 2, 2), (1, 0, 1, 3, 1), (0, 0, 0, 6, 0)
	7	(3, 0, 4, 0, 0), (2, 3, 0, 0, 2), (2, 2, 1, 1, 1), (2, 1, 2, 2, 0), (1, 2, 0, 4, 0)
	8	(3, 3, 2, 0, 0), (2, 4, 0, 2, 0)
	9	(3, 6, 0, 0, 0)
(3,6,1)	3	(1, 0, 1, 0, 1), (0, 0, 0, 3, 0)
	4	(1, 2, 0, 1, 0)
(4,8,0)	4	(1, 1, 0, 0, 2), (1, 0, 1, 1, 1), (0, 0, 0, 4, 0)

	5	(2, 1, 2, 0, 0), (1, 2, 0, 2, 0)
	6	(2, 4, 0, 0, 0)
(2,9,0)	3	(0, 0, 1, 0, 2)
	4	(1, 0, 3, 0, 0), (0, 2, 0, 1, 1), (0, 1, 1, 2, 0)
	5	(1, 3, 1, 0, 0)
(2,12,0)	4	(0, 0, 2, 0, 2)
	5	(1, 0, 4, 0, 0), (0, 3, 0, 0, 2), (0, 2, 1, 1, 1), (0, 1, 2, 2, 0)
	6	(1, 3, 2, 0, 0), (0, 4, 0, 2, 0)
	7	(1, 6, 0, 0, 0).

From this we deduce $\mathsf{c}_{\mathrm{ad}}(S) = 3$.
In order to compute the equal catenary degree, we consider only the atoms which are in the monoid of equal-length relations:

$$((0,1,0,0,1),(0,0,1,1,0)),\ ((1,0,1,0,1),(0,0,0,3,0)),\ ((1,1,0,0,2),(0,0,0,4,0))$$
$$((1,0,2,0,0),(0,1,0,2,0)),\ ((1,0,3,0,0),(0,2,0,1,1)),\ ((1,0,4,0,0),(0,3,0,0,2)).$$

Next, we compute the elements involved in the atoms and find

$$(1,5,1),\ (3,6,1),\ (4,8,0),\ (2,6,0),\ (2,9,0),\ (2,12,0).$$

Now we compute all factorizations for each of these elements and their decompositions into $\mathcal{R}$-equal classes.

(1,5,1)	2	[(0, 1, 0, 0, 1)] [(0, 0, 1, 1, 0)]	1 R-class
(3,6,1)	3	[(1, 0, 1, 0, 1)] [(0, 0, 0, 3, 0)]	2 R-classes
	4	[(1, 2, 0, 1, 0)]	1 R-class
(4,8,0)	4	[(1, 1, 0, 0, 2), (1, 0, 1, 1, 1), (0, 0, 0, 4, 0)]	1 R-class
	5	[(2, 1, 2, 0, 0), (1, 2, 0, 2, 0)]	1 R-class
	6	[(2, 4, 0, 0, 0)]	1 R-class
(2,6,0)	2	[(0, 0, 0, 0, 2)]	1 R-class
	3	[(1, 0, 2, 0, 0)] [(0, 1, 0, 2, 0)]	2 R-classes
	4	[(1, 3, 0, 0, 0)]	1 R-class
(2,9,0)	3	[(0, 0, 1, 0, 2)]	1 R-class
	4	[(1, 0, 3, 0, 0), (0, 1, 1, 2, 0), (0, 2, 0, 1, 1)]	1 R-class
	5	[(1, 3, 1, 0, 0)]	1 R-class
(2,12,0)	4	[(0, 0, 2, 0, 2)]	1 R-class
	5	[(1, 0, 4, 0, 0), (0, 2, 1, 1, 1), (0, 3, 0, 0, 2), (0, 1, 2, 2, 0)]	1 R-class
	6	[(1, 3, 2, 0, 0), (0, 4, 0, 2, 0)]	1 R-class
	7	[(1, 6, 0, 0, 0)]	1 R-class

From this we deduce $\mathsf{c}_{\mathrm{eq}}(S) = 3$. Now we find $\mathsf{c}_{\mathrm{mon}}(\mathbb{F}_2[X^2, X^3]) = \mathsf{c}_{\mathrm{mon}}(S) = 3$.

2.3.5.3. $\mathbb{F}_2[X^2, X^5]$.

The results in this case differ slightly from then ones we obtained above. We have $\#G = 2$, say $G = \{\mathbf{0}, \mathbf{e}\}$. Again, we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}^2\}$. Now we apply [**14**, Theorem 3.7.1] as before and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by

$\mathcal{A}(T) = \{X^n g \mid n \in \{2,5\},\, g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2,5\}$ and $g \in G$.
Now we apply the ACA2, see Algorithm 2.3.3, as before and find

$$\mathcal{A}(\mathcal{B}(G,T,\iota)) = \{(\mathbf{0},1), (\mathbf{e}^2,1), (1, X^2\mathbf{0}), (1, X^5\mathbf{0}), (\mathbf{e}, X^2\mathbf{e}), (\mathbf{e}, X^5\mathbf{e})\}\,.$$

Using the construction from the beginning of subsection 2.3.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G,T,\iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1,0,0,\bar{0}), (0,2,0,\bar{0}), (0,0,2,\bar{1}), (0,0,5,\bar{1}), (0,1,2,\bar{1}), (0,1,5,\bar{1})\}\,.$$

Since the atom $(1,0,0,\bar{0})$ is prime, we can use the same arguments as in Lemma 1.2.16.4 and restrict on a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with the following set of atoms

$$\mathcal{A}(\bar{S}) = \{(2,0,\bar{0}), (0,2,\bar{1}), (0,5,\bar{1}), (1,2,\bar{1}), (1,5,\bar{1})\}\,.$$

Since, in this case, step 2.3.4.1 can be performed without any bound, we compute all atoms. Now, we find a list of 25 atoms after step 2.3.4.2. Given this list, we immediately find $\rho(\mathbb{F}_2[X^2,X^5]) = 3$ in step 2.3.4.3. Now we proceed with step 2.3.4.4 and obtain $\mathsf{t}(S) = 4$, $\mathsf{c}(\mathbb{F}_2[X^2,X^5]) = 5$, and $\mathsf{D}(\mathbb{F}_2)+1+\mathsf{t}(S) = 7 \geq \mathsf{t}(\mathbb{F}_2[X^2,X^5]) \geq \max\{\mathsf{t}(S), \mathsf{c}(\mathbb{F}_2[X^2,X^5])\} = 5$. Next, we compute the monotone catenary degree. For this, we proceed as in step 2.3.4.6 and start with the adjacent catenary degree. We find $\mathsf{c}_{\mathrm{ad}}(S) = 5$. Next we compute the equal catenary degree and find $\mathsf{c}_{\mathrm{eq}}(S) = 6$. Now we find $\mathsf{c}_{\mathrm{mon}}(\mathbb{F}_2[X^2,X^5]) = \mathsf{c}_{\mathrm{mon}}(S) = 6 > 5 = \mathsf{c}(\mathbb{F}_2[X^2,X^5])$.

CHAPTER 3

Applications to special non-principal orders in algebraic number fields

3.1. Half-factorial orders in algebraic number fields and their localizations

3.1.1. Monoid-theoretic preliminaries.

DEFINITION 3.1.1. A monoid H is called *finitely primary* if there exist s, $k \in \mathbb{N}$ and a factorial monoid $F = [p_1, \ldots, p_s] \times F^\times$ with the following properties:

- $H \setminus H^\times \subset p_1 \cdot \ldots \cdot p_s F$,
- $(p_1 \cdot \ldots \cdot p_s)^k F \subset H$, and
- $(p_1 \cdot \ldots \cdot p_s)^i F \not\subset H$ for $i \in [0, k)$.

If this is the case, then we call H a *finitely primary monoid* of *rank* s and *exponent* k.
Note that this definition is slightly more restrictive than the one given in [**14**, Definition 2.9.1]. By [**14**, Theorem 2.9.2.1], we get $F = \widehat{H}$, and therefore $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times \subset \mathsf{q}(H)$.
Then, for $i \in [1, s]$, we denote by $\mathsf{v}_{p_i} : \mathsf{q}(H) \to \mathbb{Z}$ the p_i-adic valuation of $\mathsf{q}(H)$.
Now let $H \subset \widehat{H} = [p] \times \widehat{H}^\times$ be a finitely primary monoid of rank 1 and exponent k. Then we set $\mathcal{U}_i(H) = \{u \in \widehat{H}^\times \mid p^i u \in H\}$ for $i \in \mathbb{N}_0$.

As a first observation, we find

$$\mathcal{U}_i(H) = \begin{cases} H^\times & i = 0 \\ \widehat{H}^\times & i \geq k \end{cases} \quad \text{and} \quad \mathcal{U}_i(H)\mathcal{U}_j(H) \subset \mathcal{U}_{i+j}(H) \text{ for all } i, j \in \mathbb{N}_0.$$

DEFINITION 3.1.2. Let $s \in \mathbb{N}$, $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$, $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$, and $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times$ be a finitely primary monoid of rank s and exponent $\max\{k_1, \ldots, k_s\}$. Then H is a *monoid of type* $(\mathbf{e}, \mathbf{k})$ if

- $\mathsf{v}_{p_i}(H) = e_i \mathbb{N}_0 \cup \mathbb{N}_{\geq k_i}$ for all $i = [1, s]$,
- $p_1^{k_1} \cdot \ldots \cdot p_s^{k_s} \widehat{H} \subset H$.

LEMMA 3.1.3. *Let $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times$ be a reduced finitely primary monoid of rank s and exponent k.*

1. *The following statements are equivalent:*
   (a) *H is half-factorial.*
   (b) *H is of rank 1 and $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$.*
   (c) *H is of rank 1 and $(\mathcal{U}_1(H))^l = \mathcal{U}_l(H)$ for all $l \in \mathbb{N}$.*
   If any of these conditions hold, then $\mathcal{A}(H) = \{p_1 \varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$, $(\mathcal{U}_1(H))^k = \widehat{H}^\times$, and H is a monoid of type $(1, k)$.
2. *If H is a half-factorial monoid of type $(1, k)$ and $a_1, \ldots, a_{k+1}, b \in \mathcal{A}(H)$, then there are some $b_1, \ldots, b_k \in \mathcal{A}(H)$ such that $a_1 \cdot \ldots \cdot a_{k+1} = b b_1 \cdot \ldots \cdot b_k$.*
   In particular, $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) \leq \mathsf{t}(H) \leq k + 1$.

PROOF.

1. **(a) ⇒ (b).** If H is of rank $s \ge 2$, then we find $\rho(H) = \infty$ by [**14**, Theorem 3.1.5.2 (b)]. Thus H is of rank 1. Now we prove $\#\mathsf{v}_{p_1}(\mathcal{A}(H)) = 1$. [Then the assertion follows since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{n\}$ with $n \ge 2$ implies $\mathsf{v}_{p_1}(H) = n\mathbb{N}_0 \not\supset \mathbb{N}_{\ge k}$, a contradiction.] Suppose $\#\mathsf{v}_{p_1}(\mathcal{A}(H)) > 1$. Let $n = \min \mathsf{v}_{p_1}(\mathcal{A}(H))$, $m \in \mathsf{v}_{p_1}(\mathcal{A}(H)) \setminus \{n\}$, and $\varepsilon, \eta \in \widehat{H}^\times$ be such that $p_1^n\varepsilon$, $p_1^m\eta \in \mathcal{A}(H)$. Now we find
$$(p_1^m\eta)^k = (p_1^n\varepsilon)^k(p_1^{(m-n)k}\varepsilon^{-k}\eta^k).$$
On the left side there are k atoms and on the right side at least $k+1$—a contradiction to H half-factorial.
(b) ⇒ (a). Since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$, we have $\mathsf{L}(a) = \{\mathsf{v}_{p_1}(a)\}$, i.e. $\#\mathsf{L}(a) = 1$, for all $a \in H \setminus H^\times$. Therefore, H is half-factorial.
(b) ⇒ (c). Since $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$, we have $\mathcal{A}(H) = \{p_1u \mid u \in \mathcal{U}_1(H)\}$. Thus, for all $l \in \mathbb{N}$, we have $\mathcal{U}_l(H) \subset (\mathcal{U}_1(H))^l$. Since we always have $(\mathcal{U}_1(H))^l \subset \mathcal{U}_l(H)$, the assertion follows.
(c) ⇒ (b). Let $l \in \mathbb{N}_{\ge 2}$ and let $\varepsilon \in \mathcal{U}_l(H)$. By assumption, there are $\varepsilon_1, \ldots, \varepsilon_l \in \mathcal{U}_1(H)$ such that $(p_1\varepsilon_1) \cdot \ldots \cdot (p_1\varepsilon_l) = p_1^l\varepsilon$, and therefore $p_1^l\varepsilon \notin \mathcal{A}(H)$; thus $\mathsf{v}_{p_1}(\mathcal{A}(H)) = \{1\}$.
Now we prove the additional statement. $\mathcal{A}(H) = \{p_1\varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$ has already been shown and $(\mathcal{U}_1(H))^k = \mathcal{U}_k(H) = \widehat{H}^\times$ is obvious. The last statement follows immediately by considering the definition of a monoid of type $(1,k)$; see Definition 3.1.2.
2. Let $H \subset [p_1] \times \widehat{H}^\times = \widehat{H}$ be a half-factorial monoid of type $(1,k)$ and let $a_1, \ldots, a_{k+1}, b \in \mathcal{A}(H)$. By part 1, we have $\mathcal{A}(H) = \{p_1\varepsilon \mid \varepsilon \in \mathcal{U}_1(H)\}$. Then there are $\varepsilon_1, \ldots, \varepsilon_{k+1}, \eta \in \mathcal{U}_1(H)$ such that $a_i = p_1\varepsilon_i$ for $i \in [1, k+1]$ and $b = p_1\eta$. Now we find
$$a_1 \cdot \ldots \cdot a_{k+1} = (p_1\varepsilon_1) \cdot \ldots \cdot (p_1\varepsilon_{k+1}) = (p_1\eta)(p_1^k\eta^{-1}\varepsilon_1 \cdot \ldots \cdot \varepsilon_{k+1}).$$
By part 1, $(\mathcal{U}_1(H))^k = \widehat{H}^\times$, and thus there are $\eta_1, \ldots, \eta_k \in \mathcal{U}_1(H)$ such that $\eta^{-1}\varepsilon_1 \cdot \ldots \cdot \varepsilon_{k+1} = \eta_1 \cdot \ldots \cdot \eta_k$. Now we finish the proof by setting $b_i = p_1\eta_i$ for $i \in [1, k]$. □

The result of Lemma 3.1.3.2 is sharp as the following example shows.

EXAMPLE 3.1.4. Let $H \subset \widehat{H} = [p] \times \widehat{H}^\times$ be a half-factorial reduced finitely primary monoid of rank 1 and exponent $k-1$, with $k \ge 2$, such that $\widehat{H}^\times = \mathsf{C}_k^2 = \langle e_1 \rangle \times \langle e_2 \rangle$ and $\mathcal{U}_1(H) = \{1, e_1, e_2\}$.
Then $\mathsf{c}(H) = k$.

PROOF. By Lemma 3.1.3.2 we find $\mathsf{c}(H) \le k$; thus the assertion follows from the equations
$$(pe_1)^k = (pe_2)^k = p^k \quad \text{and} \quad e_1^k = e_2^k = 1 \quad \text{and} \quad \operatorname{ord}(e_1) = \operatorname{ord}(e_2) = k,$$
since one cannot construct any shorter steps in between because of the minimality of the order of e_1 respectively e_2. □

3.1.2. Locally half-factorial orders.

Definition 3.1.5 (cf. [**14**, Definition 3.6.3]). Let D be an atomic monoid.

1. If $H \subset D$ is an atomic submonoid, then we define
$$\rho(H,D) = \sup\left\{ \frac{\min \mathsf{L}_H(a)}{\min \mathsf{L}_D(a)} \middle| a \in H \setminus D^\times \right\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$
2. Let $H \subset D$ be a submonoid and $G_0 = \{[u]_{D/H} | u \in \mathcal{A}(D)\} \subset \mathsf{q}(D/H)$. We say that $H \subset D$ is *faithfully saturated* if H is atomic, $H \subset D$ is saturated and cofinal, $\rho(H,D) < \infty$, and $\mathsf{D}(G_0) < \infty$.

Lemma 3.1.6. *Let D be a half-factorial monoid and $H \subset D$ an atomic saturated submonoid.*
Then $\rho(H,D) \leq 1$.

Proof. Let $\varepsilon \in D^\times \cap H$. Then $\varepsilon \mid 1$ in D, and thus $\varepsilon \mid 1$ in H, and therefore $\varepsilon \in H^\times$. Now we find $\rho(H,D) \leq \rho(D) = 1$, by [**14**, Proposition 3.6.6]. □

Lemma 3.1.7. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D}_i^\times$ be reduced finitely primary monoids such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group, and let G be finite. Then*

1. *D is a reduced BF-monoid.*
2. *$H \subset D$ is a faithfully saturated submonoid and H is also a reduced BF-monoid.*

Proof.

1. Since D is the direct product of reduced BF-monoids, D is a reduced BF-monoid.
2. Since, by part 1, D is a reduced BF-monoid, H is a reduced BF-monoid by [**14**, Proposition 3.4.5.5]. Since G and r are finite, $H \subset D$ is faithfully saturated by [**14**, Theorem 3.6.7]. □

The following lemma offers a refinement of [**14**, Theorem 3.6.4] for faithfully saturated submonoids $H \subset D$ such that $\rho(H,D) = 1$.

Lemma 3.1.8. *Let D be a reduced atomic half-factorial monoid, $H \subset D$ a faithfully saturated submonoid with $\rho(H,D) = 1$, $G = \mathsf{q}(D/H)$ its class group, let $\mathsf{D} = \mathsf{D}(G)$ be its Davenport constant, and let each class in G contain some $u \in \mathcal{A}(D)$.*
Then
$$\mathsf{c}(H) \leq \max\left\{ \left\lfloor \frac{(\mathsf{D}+1)}{2} \mathsf{c}(D) \right\rfloor, \mathsf{D}^2 \right\}.$$

Proof. We start by developing the same machinery to compare the factorizations in H with those in D as in [**14**, Proof of Theorem 3.6.4]. Let $\pi_H : \mathsf{Z}(H) \to H$ and $\pi_D : \mathsf{Z}(D) \to D$ be the factorization homomorphisms and let $Y = \pi_D^{-1}(H) \subset \mathsf{Z}(D)$. Let $f : \mathsf{Z}(D) \to D/H$ be defined by $f(z) = [\pi_D(z)]_{D/H}$. Then f is an epimorphism and $Y = f^{-1}(0)$. Now [**14**, Proposition 2.5.1] implies that $Y \subset \mathsf{Z}(D)$ is saturated, that Y is a Krull monoid, and that f induces an isomorphism $f^* : \mathsf{Z}(D)/Y \to D/H$, since $Y \subset \mathsf{Z}(D)$ is cofinal. By [**14**, Theorem 3.4.10.5], we have $\mathsf{c}(Y) \leq \mathsf{D}$, and by [**14**, Proposition 3.4.5.3] it follows that $|v| \leq \mathsf{D}$ for all $v \in \mathcal{A}(Y)$. If $v \in Y$, then there exists a factorization $y \in \mathsf{Z}_H(\pi_D(v))$ such that $|y| \leq |v|$.

If $\tilde{z} \in Y$ and $z \in \mathsf{Z}(H)$, then we say that z is induced by $\tilde{z}$ if $z = z_1 \cdot \ldots \cdot z_m$ and $\tilde{z} = \tilde{z}_1 \cdot \ldots \cdot \tilde{z}_m$, where $\tilde{z}_j \in \mathcal{A}(Y) \subset \mathsf{Z}(D)$, $z_j \in \mathsf{Z}_H(\pi_D(\tilde{z}_j))$ and $|z_j| \le |\tilde{z}_j|$ for all $j \in [1, m]$. If z is induced by $\tilde{z}$, then $\pi_H(z) = \pi_D(\tilde{z})$ and $|z| \le |\tilde{z}|$. By definition, every factorization $\tilde{z} \in Y$ induces some factorization $z \in \mathsf{Z}(H)$. Also, if z is induced by $\tilde{z}$ and z' is induced by $\tilde{z}'$, then zz' is induced by $\tilde{z}\tilde{z}'$.

If $x = u_1 \cdot \ldots \cdot u_m \in \mathsf{Z}(H)$, where $u_j \in \mathcal{A}(H)$ and $\tilde{u}_j \in \mathsf{Z}_D(u_j)$, then $\tilde{u}_j \in \mathcal{A}(Y)$ and $|\tilde{u}_j| \le \mathsf{D}$ for all $j \in [1, m]$ by [**14**, Proposition 3.4.5.3]. Hence x is induced by $\tilde{x} = \tilde{u}_1 \cdot \ldots \cdot \tilde{u}_m$, and $|\tilde{x}| \le \mathsf{D}|x|$.

We prove the following assertions:

A0 Let $\tilde{z} \in Y$ with $\tilde{z} = a_1 \cdot \ldots \cdot a_m b_1 \cdot \ldots \cdot b_n$, where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}(H)$, $[a_1]_{D/H} = \ldots = [a_m]_{D/H} = \mathbf{0}_{D/H}$, and $[b_1]_{D/H}, \ldots, [b_n]_{D/H} \ne \mathbf{0}_{D/H}$. For any $z \in \mathsf{Z}(H)$ such that z is induced by $\tilde{z}$, we have $|z| = m + \lfloor \frac{n}{2} \rfloor$.

A1 For any $\tilde{z}, \tilde{z}' \in Y$, there exist $z, z' \in \mathsf{Z}(H)$ such that z is induced by $\tilde{z}$, z' is induced by $\tilde{z}'$, and $\mathsf{d}(z, z') \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\tilde{z}, \tilde{z}') \right\rfloor$.

A2 If $a \in H$, $\tilde{z} \in Y$, and $z, z' \in \mathsf{Z}_H(a)$ are both induced by $\tilde{z}$, then there exists a D^2-chain of factorizations in $\mathsf{Z}_H(a)$ concatenating z and z'.

Proof of A0. Let $\tilde{z} \in Y$ with $\tilde{z} = a_1 \cdot \ldots \cdot a_m b_1 \cdot \ldots \cdot b_n$, where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}(H)$, $[a_1]_{D/H} = \ldots = [a_m]_{D/H} = 0$, and $[b_1]_{D/H} = \ldots = [b_n]_{D/H} \ne 0_{D/H}$. Let now $z \in \mathsf{Z}(H)$ be induced by $\tilde{z}$. We have $a_i \in \mathcal{A}(H)$ for all $i \in [1, m]$ and—after renumbering if necessary—$b_1 \cdot \ldots \cdot b_{j_1}, b_{j_1+1} \cdot \ldots \cdot b_{j_2}, \ldots, b_{j_{k-1}+1} \cdot \ldots \cdot b_{j_k} \in \mathcal{A}(H)$ for some $k \in \mathbb{N}$ and $1 < j_1 + 1 < j_2 < j_2 + 1 < \ldots < j_{k-1} + 1 < j_k < n$ such that $a_1 \cdot \ldots \cdot a_m (b_1 \cdot \ldots \cdot b_{j_1})(b_{j_1+1} \cdot \ldots \cdot b_{j_2}) \cdot \ldots \cdot (b_{j_{k-1}+1} \cdot \ldots \cdot b_{j_k}) = z$. Then we have $|z| = m + \lfloor \frac{n}{2} \rfloor$. $\square$

Proof of A1. Suppose that $\tilde{z}, \tilde{z}' \in Y$, $\tilde{w} = \gcd(\tilde{z}, \tilde{z}') \in \mathsf{Z}(D)$, $\tilde{z} = \tilde{w}\tilde{y}$, and $\tilde{z}' = \tilde{w}\tilde{y}'$, where $\tilde{y}, \tilde{y}' \in \mathsf{Z}(D)$. By [**14**, Proposition 3.4.5.6], there exists some $\tilde{w}_0 \in \mathsf{Z}(D)$ such that $\tilde{w}_0 \mid \tilde{w}$, $\tilde{w}_0\tilde{y} \in Y$, and $|\tilde{w}_0| \le (\mathsf{D} - 1)|\tilde{y}|$. We may assume that there is no $a \in \mathcal{A}(D)$ with $a \mid \tilde{w}_0$ and $[a]_{D/H} = 0$. We set $\tilde{w}_1 = \tilde{w}_0^{-1}\tilde{w}$. Since $\tilde{z} = \tilde{w}_1(\tilde{w}_0\tilde{y}) \in Y$ and $\tilde{w}_0\tilde{y} \in Y$, we obtain $\tilde{w}_1 \in Y$, and since $\tilde{z}' = \tilde{w}_1(\tilde{w}_0\tilde{y}') \in Y$ it follows that $\tilde{w}_0\tilde{y}' \in Y$. Let $v, u, u' \in \mathsf{Z}(H)$ be such that v is induced by $\tilde{w}_0^{-1}\tilde{w}$, u is induced by $\tilde{w}_0\tilde{y}$ and u' is induced by $\tilde{w}_0\tilde{y}'$. Then $z = uv$ is induced by $\tilde{w}\tilde{y} = \tilde{z}$, $z' = u'v$ is induced by $\tilde{w}\tilde{y}' = \tilde{z}'$, and, by part **A1**,

$$\mathsf{d}(z, z') \le \max\{|u|, |u'|\} \le \max\{|\tilde{y}|, |\tilde{y}'|\} + \left\lfloor \frac{|\tilde{w}_0|}{2} \right\rfloor \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\tilde{z}, \tilde{z}') \right\rfloor. \qquad \square$$

Proof of A2. For every $\tilde{v} \in \mathcal{A}(Y)$, we fix a factorization $\tilde{v}^* \in \mathsf{Z}(H)$ which is induced by $\tilde{v}$, and, for $\bar{y} = \tilde{v}_1 \cdot \ldots \cdot \tilde{v}_s \in \mathsf{Z}(Y)$, we set $\bar{y}^* = \tilde{v}_1^* \cdot \ldots \cdot \tilde{v}_s^* \in \mathsf{Z}(H)$. Then $\bar{y}^*$ is induced by $\pi_Y(\bar{y})$, $|\bar{y}^*| \le |\pi_Y(\bar{y})| \le \mathsf{D}|\bar{y}|$, and if $\bar{y}_1, \bar{y}_2 \in \mathsf{Z}(Y)$, then $\mathsf{d}(\bar{y}_1^*, \bar{y}_2^*) \le \left\lfloor \frac{\mathsf{D}+1}{2} \mathsf{d}(\bar{y}_1, \bar{y}_2) \right\rfloor$ by **A1**. Let now $z, z' \in \mathsf{Z}_H(a)$ be both induced by $\tilde{z}$. Then $\tilde{z} = \tilde{v}_1 \cdot \ldots \cdot \tilde{v}_r = \tilde{v}_1' \cdot \ldots \cdot \tilde{v}_{r'}'$, $z = v_1 \cdot \ldots \cdot v_r$, and $z' = v_1' \cdot \ldots \cdot v_{r'}'$, where $\tilde{v}_i, \tilde{v}_i' \in \mathcal{A}(Y)$, v_i is induced by $\tilde{v}_i$, and v_i' is induced by $\tilde{v}_i'$. Since $\bar{y} = \tilde{v}_1 \cdot \ldots \cdot \tilde{v}_r \in \mathsf{Z}_Y(\tilde{z})$, $\bar{y}' = \tilde{v}_1' \cdot \ldots \cdot \tilde{v}_{r'}' \in \mathsf{Z}_Y(\tilde{z})$, and $\mathsf{c}(Y) \le \mathsf{D}$, there exists a D-chain $\bar{y} = \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_l = \bar{y}'$ in $\mathsf{Z}_Y(\tilde{z})$ concatenating $\bar{y}$ and $\bar{y}'$ in $\mathsf{Z}_Y(\tilde{z})$. Then $\bar{y}_0^*, \bar{y}_1^*, \ldots, \bar{y}_l^*$ is a D-chain in $\mathsf{Z}_H(a)$ concatenating $\bar{y}^*$ and $\bar{y}'^*$. We have $\bar{y}_0^* = \tilde{v}_1^* \cdot \ldots \cdot \tilde{v}_r^*$, $z = v_1 \cdot \ldots \cdot v_r$, and since both v_i and v_i^* are induced by $\tilde{v}_i$, it follows that $\max\{|v_i|, |v_i^*|\} \le |\tilde{v}_i| \le \mathsf{D}$. For $i \in [0, r]$, we set $z_i = \tilde{v}_1^* \cdot \ldots \cdot \tilde{v}_i^* v_{i+1} \cdot \ldots \cdot v_r \in \mathsf{Z}_H(a)$. Then $z = z_0, z_1, \ldots, z_r = \bar{y}^*$ is a

D^2-chain concatenating z and $\bar{y}^*$. In the same way, we get a D-chain concatenating $\bar{y}'^*$ and z'. Connecting these three chains, we get a D^2-chain in $\mathsf{Z}_H(a)$ concatenating z and z'. □

Suppose that $a \in H$ and $z,\, z' \in \mathsf{Z}_H(a)$. Let $\tilde{z},\, \tilde{z}' \in Y$ be such that z is induced by $\tilde{z}$ and z' is induced by $\tilde{z}'$. Then $\tilde{z},\, \tilde{z}' \in \mathsf{Z}_D(a)$, and therefore there exists a $\mathsf{c}(D)$-chain $\tilde{z} = \tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_l = \tilde{z}'_l$ in $\mathsf{Z}_D(a)$. For $i \in [0, l-1]$, **A1** gives the existence of factorizations $z'_i,\, z''_i \in \mathsf{Z}_H(a)$ such that z'_i is induced by $\tilde{z}_i$, z''_i is induced by $\tilde{z}_{i+1}$, and $\mathsf{d}(z'_i, z''_i) \le \left\lfloor \frac{\mathsf{D}+1}{2}\mathsf{c}(D) \right\rfloor$. By **A2**, there exist D^2-chains of factorizations in $\mathsf{Z}_H(a)$ concatenating z and z'_0, z''_i and z'_{i+1} for all $i \in [0, l-1]$, and z_{l-1} and z'. Connecting all these chains, we obtain a $\max\left\{ \left\lfloor \frac{(\mathsf{D}+1)}{2}\mathsf{c}(D) \right\rfloor, \mathsf{D}^2 \right\}$-chain concatenating z and z'. □

Lemma 3.1.9. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D}_i^\times$ be reduced half-factorial but not factorial monoids of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group, and let G be finite such that each class in G contain some $p \in P$.*
Then

1. *$2 \le \mathsf{c}(D) = \max\{\mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\} \le \max\{k_1, \ldots, k_r\} + 1 \le 3$ and D is half-factorial.*
   In particular, $\mathsf{c}(D) = 2$ and $\mathsf{t}(D) = 2$, if $k_1 = \ldots = k_r = 1$.
2. *If $\#G = 1$, then $\mathsf{c}(H) = \mathsf{c}(D)$, $\mathsf{t}(H) = \mathsf{t}(D)$, and H is half-factorial.*
3. *If $\#G \ge 3$, then $(\mathsf{D}(G))^2 \ge \mathsf{c}(H) \ge 3$ and $\min \triangle(H) = 1$.*
4. *If $\#G = 2$, then $\mathsf{c}(H) \le 4$ and $\rho(H) \le 2$.*

Proof.

1. By Lemma 3.1.7.1, D is atomic. Trivially, we have $\mathsf{c}(\mathcal{F}(P)) = 0$. By Lemma 3.1.3.2 and the fact that D_i is not factorial, we find $2 \le \mathsf{c}(D_i) \le k_i + 1 \le 3$ for all $i \in [1, r]$. By Lemma 1.2.18.3, we find
$$\begin{aligned}\mathsf{c}(D) &= \max\{\mathsf{c}(\mathcal{F}(P)), \mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\}\\ &= \max\{\mathsf{c}(D_1), \ldots, \mathsf{c}(D_r)\}\\ &= \max\{k_1, \ldots, k_r\} + 1 \le 3.\end{aligned}$$
   Thus the first part of the assertion follows. Since D is the direct product of half-factorial monoids, D is half-factorial by Lemma 1.2.18.4. We have $\mathsf{t}(D_i) = 2$ if $k_i = 1$ for all $i \in [1, r]$ by Lemma 3.1.3.2. Now $\mathsf{t}(D) = 2$ follows by Lemma 1.2.18.6.
2. Here we have $H = D$ and thus the assertion follows from part 1.

Before the proof of the two remaining parts we make the following observations. By Lemma 3.1.7.2, H is atomic, $H \subset D$ is a faithfully saturated submonoid, and, by Lemma 3.1.6, we have $\rho(H, D) \le 1$.

3. By part 1, we have $\mathsf{c}(D) \le 3$, by Lemma 1.2.17.1, we have $\min \triangle(H) = 1$, and, by [**14**, Lemma 1.4.9.2], we have $\mathsf{D}(G) \ge 3$. Using [**14**, Theorem 3.6.4.1], we find
$$3 \le \mathsf{D}(G) \le \mathsf{c}(H) \le \rho(H, D) \max\{\mathsf{c}(D), \mathsf{D}(G)\}\mathsf{D}(G) = (\mathsf{D}(G))^2.$$
4. Since $\#G = 2$, we have $\mathsf{D}(G) = 2$, and since $D_1 \times \ldots \times D_r$ is half-factorial, i.e. $\rho(D_1 \times \ldots \times D_r) = 1$, we find $\rho(H) \le 2$ by Lemma 1.2.17.2. When we apply

Lemma 3.1.8, we find

$$\mathsf{c}(H) \le \max\left\{\left\lfloor (\mathsf{D}(G)+1)\frac{\mathsf{c}(D)}{2}\right\rfloor, \mathsf{D}(G)^2\right\} \le \left\{\left\lfloor \frac{9}{2}\right\rfloor, 4\right\} = 4. \qquad \square$$

Definition 3.1.10. Let D be a monoid, $P \subset D$ a set of prime elements, and $T \subset D$ a submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H) = \mathsf{q}(D)/\mathsf{q}(H)$ its class group, $G_0 = \{[p]_{D/H} \mid p \in P\} \subset G$ the set of all classes in G containing some $p \in P$, and let $\mathcal{B}(G_0, T, \iota)$ be the T-block monoid over G_0 defined by the homomorphism $\iota : T \to G$, $\iota(t) = [t]_{D/H}$.
For a subset $S \subset \mathcal{B}(G_0, T, \iota)$ and an element $g \in G_0$, we set $S_g = S \cap \iota^{-1}(\{g\})$.

Lemma 3.1.11. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D_i}^\times$ be reduced half-factorial monoids of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group with $\#G = 2$, say $G = \{\mathbf{0}, g\}$, let each class in G contain some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$.*
Then we find the following for the atoms of the $(D_1 \times \ldots \times D_r)$-block monoid over G defined by ι, i.e., $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$:

$$\begin{aligned}
&\mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota))\\
&= \{\mathbf{0}, g^2\}\\
&\cup \{p_i\varepsilon \mid i \in [1, r],\ \varepsilon \in \mathcal{U}_1(D_i),\ \iota(p_i\varepsilon) = \mathbf{0}\}\\
&\cup \{p_i\varepsilon g \mid i \in [1, r],\ \varepsilon \in \mathcal{U}_1(D_i),\ \iota(p_i\varepsilon) = g\}\\
&\cup \{p_i^2\varepsilon \mid i \in [1, r],\ \varepsilon \in (\widehat{D_i}^\times)_{\mathbf{0}} \setminus (\mathcal{U}_1(D_i)_{\iota(p_i)})^2\}\\
&\cup \{p_ip_j\varepsilon_i\varepsilon_j \mid i, j \in [1, r],\ i \ne j,\ \varepsilon_i \in \mathcal{U}_1(D_i),\ \varepsilon_j \in \mathcal{U}_1(D_j),\ \iota(p_i\varepsilon_i) = \iota(p_j\varepsilon_j) = g\}\,.
\end{aligned}$$

Proof. For short, we write $\mathcal{B} = \mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$. Since $\#G = 2$, we have $\mathsf{D}(G) = 2$, and thus every atom of $\mathcal{B}$ is a product of at most two atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$. First, we write down all atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$, namely

$$\mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) = \{\mathbf{0}, g\} \cup \bigcup_{i \in [1,r]} \{p_i\varepsilon \mid \varepsilon \in \mathcal{U}_1(D_i)\}\,,$$

by Lemma 3.1.3.1. Now, we find

$$\begin{aligned}
&\mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) \cap \mathcal{B} = \{\mathbf{0}\} \cup \{p_i\varepsilon \mid i \in [1, r],\ \varepsilon \in \mathcal{U}_1(D_i),\ \iota(p_i\varepsilon) = \mathbf{0}\}\,, \text{ and}\\
&\mathcal{A}(\mathcal{F}(G) \times D_1 \times \ldots \times D_r) \setminus \mathcal{B} = \{\mathbf{g}\} \cup \{p_i\varepsilon \mid i \in [1, r],\ \varepsilon \in \mathcal{U}_1(D_i),\ \iota(p_i\varepsilon) = g\}\,.
\end{aligned}$$

By Lemma 3.1.7, D and H are reduced, and therefore $\varepsilon_i\varepsilon_j \notin \mathcal{B}$ for all $i, j \in [1, r]$, $i \ne j$, $\varepsilon_i \in \mathcal{U}_1(D_i)$, and $\varepsilon_j \in \mathcal{U}_1(D_i)$. Thus the following products of two atoms of $\mathcal{F}(G) \times D_1 \times \ldots \times D_r$ are atoms of $\mathcal{B}$:

$$\begin{aligned}
\mathcal{A}(\mathcal{B}) \supset &\{g^2\}\\
&\cup\{p_i\varepsilon g \mid i \in [1, r],\ \varepsilon \in \mathcal{U}_1(D_i),\ \iota(p_i\varepsilon) = g\}\\
&\cup\{p_i^2\varepsilon \mid i \in [1, r],\ \varepsilon \in (\widehat{D_i}^\times)_{\mathbf{0}} \setminus (\mathcal{U}_1(D_i)_{\iota(p_i)})^2\}\\
&\cup\{p_ip_j\varepsilon_i\varepsilon_j \mid i, j \in [1, r],\ i \ne j,\ \varepsilon_i \in \mathcal{U}_1(D_i),\ \varepsilon_j \in \mathcal{U}_1(D_j),\ \iota(p_i\varepsilon_i) = \iota(p_j\varepsilon_j) = g\}\,.
\end{aligned}$$

Since we have run through all possible combinations, the assertion follows. $\square$

Lemma 3.1.12. *Let $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$ be a monoid, where $P \subset D$ is a set of prime elements, $r \in \mathbb{N}$, and, for all $i \in [1, r]$, $D_i \subset [p_i] \times \widehat{D_i}^{\times}$ is a reduced half-factorial but not factorial monoid of type $(1, 1)$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group, $\#G = 2$, and let each class in G contain some $p \in P$. Let $\iota : D_1 \times \ldots \times D_r \to G$ be defined by $\iota(t) = [t]_{D/H}$, and denote by $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ the $(D_1 \times \ldots \times D_r)$-block monoid over G defined by ι and set $|\cdot|_{\mathcal{B}} = |\cdot|_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}$.*

1. *If $(x, y) \in \sim_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}$ with $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}}$ and $|y|_{\mathcal{B}} \geq 5$, then there is a monotone $\mathcal{R}$-chain concatenating x and y; in particular, $x \approx y$, and if $|x|_{\mathcal{B}} = |y|_{\mathcal{B}}$, then $x \approx_{\mathrm{eq}} y$.*
2. *Additionally,*

$$\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) \leq \sup\{|y|_{\mathcal{B}} \mid (x, y) \in \mathcal{A}(\sim_{\mathcal{B}(G,D_1\times\ldots\times D_r,\iota)}),\ |x|_{\mathcal{B}} \leq |y|_{\mathcal{B}} \leq 4\}.$$

Proof. Let $\#G = 2$, say $G = \{\mathbf{0}, g\}$. By Lemma 1.2.16.4, we set $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_r, \iota) \mid \mathbf{0} \nmid S\}$. Before we start the actual proof, we establish some machinery to deal with factorizations in $\mathcal{B}$ and their lengths more systematically.
We set $D_0 = \mathcal{F}(\{g\})$, whence $\mathcal{A}(D_0) = \{g\}$ and $\mathsf{Z}(D_0) = D_0$. We define $\iota : D_0 \to G$ by $\iota(g^k) = kg$ for all $k \in \mathbb{Z}$. For $i \in [0, r]$, let $\pi_i : \mathsf{Z}(D_i) \to D_i$ be the factorization homomorphism. We set $D' = D_0 \times \ldots \times D_r$ and obtain $\mathcal{A}(D') = \mathcal{A}(D_0) \cup \ldots \cup \mathcal{A}(D_r)$. If $a = a_0 \cdot \ldots \cdot a_r \in D'$, where $a_i \in D_i$ for all $i \in [0, r]$, then we set $\iota(a) = \iota(a_0) + \iota(a_1 \cdot \ldots \cdot a_r) = \iota(a_0) + \ldots + \iota(a_r)$. Then $\iota : D' \to G$ is a homomorphism and $\mathcal{B} = \iota^{-1}(\mathbf{0}) \subset D'$ is a saturated submonoid, whose atoms are given by the following assertion **A1**.

A1 An element $x \in D_0 \times \cdot \ldots \cdot \times D_r$ is an atom of $\mathcal{B}$ if and only if it is of one of the following forms:
- $x = a \in \mathcal{A}(D_i)$ for some $i \in [1, r]$ and $\iota(a) = \mathbf{0}$.
- $x = a_1 a_2$, where $a_1 \in \mathcal{A}(D_i)$, $a_2 \in \mathcal{A}(D_j)$, for some $i, j \in [0, r]$, $i \neq j$, and $\iota(a_1) = \iota(a_2) = g$.
- $x = a_1 a_2$, where $a_1, a_2 \in \mathcal{A}(D_i)$ for some $i \in [0, r]$ such that $\iota(v) = g$ for all $v \in \mathcal{A}(D_i)$.

We will call the atoms of the third form *pure in i*.

Proof of **A1**. By the listing of all atoms of $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ in Lemma 3.1.11 and the fact that $\mathcal{A}(\mathcal{B}) = \mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)) \setminus \{\mathbf{0}\}$, we must only show the last statement in the case $i \in [1, r]$. Suppose there are $a_1, a_2 \in \mathcal{A}(D_i)$ such that $a = a_1 a_2 \in \mathcal{A}(\mathcal{B})$. Then, obviously, $\iota(a_1) = \iota(a_2) = g$. Now we assume, there is some $v \in \mathcal{A}(D_i)$ with $\iota(v) = \mathbf{0}$. By Lemma 3.1.3.2, there is $v' \in \mathcal{A}(D_i)$ such that $a_1 a_2 = v v'$, and then $\iota(v') = \mathbf{0}$, a contradiction. $\square$

Let $F = \mathsf{Z}(D') = \mathsf{Z}(D_0) \times \ldots \times \mathsf{Z}(D_r) = \mathcal{F}(\mathcal{A}(D'))$ be the factorization monoid of D'. Then $\pi = \pi_0 \times \ldots \times \pi_r : F \to D'$ is the factorization homomorphism of D'. We denote by $|\cdot| = |\cdot|_F$ the length function in the free monoid F, and for $x, y \in F$ we write $x \mid y$ instead of $x \mid_F y$. For $a \in \mathcal{A}(\mathcal{B})$, let $\theta_0(a) \in \pi^{-1}(a) \subset \mathsf{Z}(D')$ be a factorization of a in D'. If $a \in \mathcal{A}(D')$, then $\theta_0(a) = a$; otherwise $\theta_0(a) = a_1 a_2 \in F$ for some $a_1, a_2 \in \mathcal{A}(D')$ such that $a = a_1 a_2$ in D'. By **A1**, $\#\pi^{-1}(a) = 1$ unless a is pure in i for some $i \in [1, r]$. Let $\theta : \mathsf{Z}(\mathcal{B}) \to F$ be the unique monoid homomorphism satisfying $\theta|\mathcal{A}(\mathcal{B}) = \theta_0$. Then θ

induces the following commutative diagram

$$\begin{tikzcd} \mathsf{Z}(\mathcal{B}) \arrow[r, "\theta"] \arrow[d, "\pi_{\mathcal{B}}"] & F = \mathsf{Z}(D') \arrow[d, "\pi_{D'}"] \\ \mathcal{B} \arrow[r, hook] & D', \end{tikzcd}$$

where $\pi_{\mathcal{B}}$ denotes the factorization homomorphism of $\mathcal{B}$ and the bottom arrow denotes the inclusion. For $x \in \mathsf{Z}(\mathcal{B})$, we set $|x| = |\theta(x)|$. For $x \in \mathsf{Z}(\mathcal{B})$, we define its components $x_i \in \mathsf{Z}(D_i)$ for $i \in [0, r]$ by $\theta(x) = x_0 \cdot \ldots \cdot x_r$. Then $\pi \circ \theta(x) \in \mathcal{B}$ implies $\iota \circ \pi_0(x_0) + \ldots + \iota \circ \pi_r(x_r) = \mathbf{0}$. For $i \in [0, r]$, we set $x_i = u_{i,1} \cdot \ldots \cdot u_{i,k_i} v_{i,1} \cdot \ldots \cdot v_{i,l_i}$, where $u_{i,j},\ v_{i,j} \in \mathcal{A}(D_i)$, $\iota(u_{i,j}) = \mathbf{0}$, and $\iota(v_{i,j}) = g$. We define $x_i',\ x_i'' \in \mathsf{Z}(D_i)$ by $x_i' = u_{i,1} \cdot \ldots \cdot u_{i,k_i}$ and $x_i'' = v_{i,1} \cdot \ldots \cdot v_{i,l_i}$, whence $x_i = x_i' x_i''$. In particular, $|x_0'| = 0$, $x_0 = x_0''$, and $\iota \circ \pi_i(x_i) = l_i g = |x_i''| g$. Therefore we obtain $|x_0''| + \ldots + |x_r''| \equiv 0 \mod 2$. If $i \in [1, r]$ and $a \in D_i$ is such that $a \mid x_i'$, then $a \mid_{\mathcal{B}} x$. In $\mathsf{Z}(\mathcal{B})$, there is a factorization $x = u_1 \cdot \ldots \cdot u_m v_1 \cdot \ldots \cdot v_n$, where $u_j,\ v_j \in \mathcal{A}(\mathcal{B})$, $|u_j| = 1$ for all $j \in [1, m]$, $|v_j| = 2$ for all $j \in [1, n]$, and we obtain

$$m = \sum_{i=1}^{r} |x_i'|, \quad n = \frac{1}{2} \sum_{i=0}^{r} |x_i''|, \quad \text{and} \quad |x|_{\mathcal{B}} = m + n = \frac{1}{2} \sum_{i=0}^{r} (|x_i| + |x_i'|) \leq \sum_{i=0}^{r} |x_i|.$$

Assume now that $x = x_0 \cdot \ldots \cdot x_r$, $y = y_0 \cdot \ldots \cdot y_r \in \mathsf{Z}(\mathcal{B})$ are as above, and suppose that $(x, y) \in \sim_{\mathcal{B}}$. Then $x_0 = y_0$, $|x_i| = |y_i|$ (since each D_i is half-factorial), $\pi_i(x_i) = \pi_i(y_i) \in D_i$, thus $\iota \circ \pi_i(x_i) = \iota \circ \pi_i(y_i) \in G$, and therefore $|x_i''| \equiv |y_i''| \mod 2$ and $|x_i'| \equiv |y_i'| \mod 2$ for all $i \in [1, r]$. Consequently, it follows that the following are all equivalent:

- $|x|_{\mathcal{B}} \leq |y|_{\mathcal{B}}$
- $\sum_{i=1}^{r} |x_i'| \leq \sum_{i=1}^{r} |y_i'|$
- $\sum_{i=1}^{r} |x_i''| \geq \sum_{i=1}^{r} |y_i''|$

Additionally, we find

$$2|x|_{\mathcal{B}} = \sum_{i=0}^{r} (|x_i| + |x_i'|) \geq \sum_{i=0}^{r} (|y_i|) \geq |y|_{\mathcal{B}},$$

and thus $|y|_{\mathcal{B}} \geq 5$ implies $|x|_{\mathcal{B}} \geq 3$.

Before we start with the actual proof of part 1 of Lemma 3.1.12, we prove the following reduction step.

A2 In the proof of part 1 of Lemma 3.1.12, we may assume that $|x_i| = |y_i| \geq 2$ for all $i \in [1, r]$.

Proof of **A2**. If $i \in [1, r]$, then $|x_i| = 0$ if and only if $|y_i| = 0$, and in this case we may neglect this component. If $|x_i| = 0$ for all $i \in [1, r]$, then there is nothing to do. If $i \in [1, r]$, then $|x_i| = 1$ if and only if $|y_i| = 1$, and then $x_i = y_i \in \mathcal{A}(D_i)$. Suppose that $i \in [1, r]$ and $|x_i| = 1$. If $\iota(x_i) = \mathbf{0}$, then $x_i \in \mathcal{A}(\mathcal{B})$ and x_i is a greatest common divisor of x and y in $\mathsf{Z}(\mathcal{B})$, hence $(x,\ y)$ is a monotone $\mathcal{R}$-chain concatenating x and y. If $\iota(x_i) = g$, then we set $\tilde{x} = g x_i^{-1} x$, $\tilde{y} = g y_i^{-1} y$, and then $(\tilde{x}, \tilde{y}) \in \sim_{\mathcal{B}}$, $|\tilde{x}_i| = |\tilde{y}_i| = 0$, and whenever there is a monotone $\mathcal{R}$-chain concatenating $\tilde{x}$ and $\tilde{y}$, then there is a monotone $\mathcal{R}$-chain concatenating x and y. $\square$

Now we are ready to do the actual proof of the lemma. Suppose that $(x, y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \geq 5$, $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}}$, $x = x_0 \cdot \ldots \cdot x_r$, $y = y_0 \cdot \ldots \cdot y_r$, $x_i = x_i' x_i''$, $y_i = y_i' y_i''$ as above, and $|x_i| = |y_i| \geq 2$ for all $i \in [0, r]$. We shall use **A1** and Lemma 3.1.3.2 again and again without mentioning this explicitly. Of course, we may assume that there is no $a \in \mathcal{A}(\mathcal{B})$ such that $a \mid_{\mathcal{B}} x$ and $a \mid_{\mathcal{B}} y$, since then there is, trivially, a monotone $\mathcal{R}$-chain concatenating x and y. For now assume $|x|_{\mathcal{B}} \geq 4$; the remaining case, where $|x|_{\mathcal{B}} = 3$, will be studied at the end of the proof after Case 3.

Case 1. There is some $i \in [1, r]$ such that $|x_i'| \geq 1$ and $|y_i'| \geq 1$.

Case 1.1. There is some $i \in [1, r]$ such that $|x_i|' \geq 2$ and $|y_i'| \geq 1$.

Let a_1, a_2, $b \in \mathcal{A}(D_i)$ be such that $a_1 a_2 \mid x_i'$ and $b \mid y_i'$. Then there is some $b' \in \mathcal{A}(D_i)$ such that $a_1 a_2 = bb'$. Then $\iota(b') = \mathbf{0}$, and if $x^* \in \mathsf{Z}(\mathcal{B})$ is such that $x = a_1 a_2 x^*$, then x, $bb'x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 1.2. There is some $i \in [1, r]$ such that $|x_i'| = 1$ and $|y_i'| \geq 1$.

Then $x_i' \in \mathcal{A}(\mathcal{B})$. Let a, $b \in \mathcal{A}(D_i)$ be such that $a \mid x_i''$ and $b \mid y_i'$. Let $u \in \mathcal{A}(F)$ be such that $au \in \mathcal{A}(\mathcal{B})$ and $au \mid_{\mathcal{B}} x$. Since $x_i' \in \mathcal{A}(D_i)$, we obtain $u \notin \mathcal{A}(D_i)$. Let $b' \in \mathcal{A}(D_i)$ be such that $x_i' a = bb'$, whence $\iota(b') = g$ and $b'u \in \mathcal{A}(\mathcal{B})$. If $x = x_i'(au)x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, then $|x^*|_{\mathcal{B}} \geq 1$, and x, $b(b'u)x^*$, y is a monotone $\mathcal{R}$-chain concatenting x and y.

Reduction 1. By Case 1, we may now assume that, for all $i \in [1, r]$, either $|x_i'| = 0$ or $|y_i'| = 0$. In particular, if $|x_i'| \geq 1$, then $|y_i'| = 0$, and therefore $|x_i'| \geq 2$, since $|x_i'| \equiv |y_i'|$ mod 2. Similarly, if $|y_i'| \geq 1$, then $|y_i'| \geq 2$.

Case 2. There is some $i \in [1, r]$ such that $|y_i'| \geq 1$.

In this case, $|x_i'| = 0$ by Reduction 1, hence $|y_i'| \geq 2$ and $|x_i''| \geq 2$. Let $b \in \mathcal{A}(D_i)$ be such that $b \mid y_i'$. Now we must distinguish a few more cases based on $|x|_{\mathcal{B}}$ and $|y|_{\mathcal{B}}$.

Case 2.1. $|x|_{\mathcal{B}} = |y|_{\mathcal{B}}$.

Note that in this case $|x|_{\mathcal{B}} = |y|_{\mathcal{B}} \geq 5$. We assert that there is some $j \in [1, r] \setminus \{i\}$ such that $|y_j'| < |x_j'|$. Indeed, if $|y_j'| \geq |x_j'|$ for all $j \in [1, r] \setminus \{i\}$, then

$$\sum_{\nu=1}^{r} |x_\nu'| \leq \sum_{\substack{\nu=1 \\ \nu \neq i}}^{r} |y_\nu'| < \sum_{\nu=1}^{r} |y_\nu'|,$$

and therefore $|x|_{\mathcal{B}} < |y|_{\mathcal{B}}$, a contradiction. By Reduction 1, we obtain $|y_j'| = 0$, hence $|y_j''| \geq 2$, and $|x_j'| \geq 2$. We write x in the form

$$x = (a_1 u_1) \cdot \ldots \cdot (a_k u_k)(a_{k+1} u_1^*) \cdot \ldots \cdot (a_{k+t} u_t^*)(e_1 u_{k+1}) \cdot \ldots \cdot (e_s u_{k+s}) \tilde{x},$$

where k, s, $t \in \mathbb{N}_0$, $x_i'' = a_1 \cdot \ldots \cdot a_{k+t}$, $x_j'' = u_1 \cdot \ldots \cdot u_{k+s}$, $u_1^*, \ldots, u_t^*$, $e_1, \ldots, e_s \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $k + t \geq 2$, and

$$\tilde{x} = (e_1 \cdot \ldots \cdot e_s)^{-1} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{r} x_\nu' \prod_{\substack{\nu=1 \\ \nu \neq i,j}}^{r} x_\nu'' \in \mathsf{Z}(\mathcal{B}).$$

Let c_1, c_2, $d_1 \in \mathcal{A}(D_j)$ be such that $c_1 c_2 \mid x_j'$, $d_1 \mid y_j''$, and choose $d_2 \in \mathcal{A}(D_j)$ such that $c_1 c_2 = d_1 d_2$, whence $\iota(d_2) = g$.

Case 2.1a. $t \geq 2$.

Choose som $b' \in \mathcal{A}(D_i)$ such that $a_{k+1} a_{k+2} = bb'$. Then $\iota(b') = \mathbf{0}$, $d_1 u_1^*$, $d_2 u_2^* \in \mathcal{A}(\mathcal{B})$, and we set $x = (a_{k+1} u_1^*)(a_{k+2} u_2^*) c_1 c_2 x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \geq 1$. Now x, $bb'(d_1 u_1^*)(d_2 u_2^*)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1b. $t = 1$.

Note that $|x_i''| = k+1 \geq 2$ implies $k \geq 1$. Assume first that there is some $v \in \mathcal{A}(\mathcal{B})$ such that $|v| = 2$ and $v \mid \tilde{x}$, say $v = v'v''$, where v', $v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$ and $\iota(v') = \iota(v'') = g$.

Then it follows that a_1v', $u_1v'' \in \mathcal{A}(\mathcal{B})$, and we set $x = (a_1u_1)(a_{k+1}v_1)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. We set $x' = (a_1v')(a_{k+1}v_1)(u_1v'')x^*$, then $x' \in \mathsf{Z}(\mathcal{B})$, $(x, x') \in \sim_{\mathcal{B}}$, and $x \approx_{\mathrm{eq}} x'$. Hence, $(x', y) \in \sim_{\mathcal{B}}$, $x' \approx_{\mathrm{eq}} y$ by Case 2.1a, and therefore $x \approx_{\mathrm{eq}} y$.
Now we set $u_1^* = u$, and we let $m \in [0, r] \setminus \{i, j\}$ be such that $u \in \mathcal{A}(D_m)$. We write x in the form

$$x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(a_{k+1}u)(e_1u_{k+1}) \cdot \ldots \cdot (e_su_{k+s}) \prod_{\substack{\nu=1\\ \nu\ne i}}^{r} x'_\nu, \quad \text{where } s+1 = \sum_{\substack{\nu=0\\ \nu\ne i,j}}^{r} |x''_\nu|.$$

We may assume that $|x'_n| = 0$ for all $n \in [1, r] \setminus \{m, j\}$. Indeed, let $n \in [1, r] \setminus \{m, j\}$ be such that $|x'_n| \ge 1$. Then $|x'_n| \ge 2$, $|y'_n| = 0$, and $|y''_n| \ge 2$. Let v_1, v_2, $w_1 \in \mathcal{A}(D_n)$ be such that $v_1v_2 \mid x'_n$ and $w_1 \mid y''_n$, and choose $b_1 \in \mathcal{A}(D_i)$ and $w_2 \in \mathcal{A}(D_n)$ such that $a_1a_{k+1} = bb_1$ and $v_1v_2 = w_1w_2$. Then $\iota(b_1) = \mathbf{0}$, $\iota(w_2) = g$, u_1w_1, $uw_2 \in \mathcal{A}(\mathcal{B})$, and if $x = (a_1u_1)(a_{k+1}u)v_1v_2x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, then $|x^*|_{\mathcal{B}} \ge 1$, and x, $bb_1(u_1w_1)(uw_2)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.
Thus suppose that $|x'_n| = 0$ for all $n \in [1, r] \setminus \{m, j\}$, and consequently

$$x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(a_{k+1}u)(e_1u_{k+1}) \cdot \ldots \cdot (e_su_{k+s})x'_jx'_m.$$

We assert that there exist v_1, v_2, $v_3 \in \mathcal{A}(D_m)$ and w_1, w_2, $w_3 \in \mathcal{A}(D_j)$ such that $v_1v_2v_3 \mid y_m$, $w_1w_2w_3 \mid y_j$, $\iota(v_\nu) = \iota(w_\nu) = g$, $v_\nu w_\nu \in \mathcal{A}(\mathcal{B})$ and $v_\nu w_\nu \mid_{\mathcal{B}} y$ for all $\nu \in [1, 3]$. Indeed, observe that

$$|y''_i| = |y_i| - |y'_i| \le |y_i| - 2 = |x_i| - 2 = |x''_i| - 2 = k - 1,$$

$$|y''_j| = |y_j| = |x'_j| + |x''_j| \ge 2 + |x''_j| = k + s + 2,$$

and set $y''_j = y_{j,1} \cdot \ldots \cdot y_{j,\mu}$, where $\mu = |y''_j|$, and, for all $\alpha \in [1, \mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1, \mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(B)$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. Since $|x'_j| \ge 1$, it follows that $u_{j,\alpha} \notin \mathcal{A}(D_j)$ for all $\alpha \in [1, \mu]$. For $\nu \in [0, r] \setminus \{j\}$, we set $N_\nu = |\{\alpha \in [1, \mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu\}|$, and we obtain

$$\mu = \sum_{\substack{\nu=0\\ \nu\ne j}}^{r} N_\nu = N_m + N_i + \sum_{\substack{\nu=0\\ \nu\ne i,j,m}}^{r} N_\nu \le N_m + |y''_i| + \sum_{\substack{\nu=0\\ \nu\ne i,j,m}}^{r} |y_\nu|.$$

Since $|y_\nu| = |x_\nu| = |x''_\nu|$ for all $\nu \in [0, r] \setminus \{i, j, m\}$ and $|x''_m| \ge 1$, it follows that

$$k+s+2 \le \mu \le N_m + k - 1 + \sum_{\substack{\nu=0\\ \nu\ne i,j,m}}^{r} |x''_\nu| \le N_m + k - 1 + \sum_{\substack{\nu=0\\ \nu\ne i,j}}^{r} |x''_\nu| - |x''_m| = N_m + k + s - 1$$

and therefore $N_m \ge 3$, which implies the existence of v_1, v_2, v_3 and w_1, w_2, w_3 as asserted. In particular, it follows that $|x_m| = |y_m| \ge |y''_m| \ge 3$ and $|x_j| = |y_j| \ge |y''_j| \ge 3$. Let $u'_1 \in \mathcal{A}(D_j)$ be such that $u_1u_{k+1} = u'_1w_1$. Then $\iota(u'_1) = g$ and $a_1u'_1 \in \mathcal{A}(\mathcal{B})$.

Case 2.1b′. $s \ge 1$.
We assume first that $|x'_m| \ge 1$. Let $u' \in \mathcal{A}(D_m)$ be such that $u' \mid x'_m$. Then there exists some $v \in \mathcal{A}(D_m)$ such that $uu' = v_1v$, hence $\iota(v) = \mathbf{0}$, and $x = (a_1u_1)(a_{k+1}u)(e_1u_{k+1})u'x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Since $a_1u'_1 \in \mathcal{A}(\mathcal{B})$ and $a_{k+1}e_1 \in \mathcal{A}(\mathcal{B})$, we conclude that x, $(a_1u'_1)(a_{k+1}e_1)(v_1w_1)vx^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.
Assume now that $|x'_m| = 0$. Then $|x''_m| = |x_m| \ge 3$, and (after renumbering if necessary) we may assume that $e_1 \in \mathcal{A}(D_m)$. Let $v \in \mathcal{A}(D_m)$ be such that $ue_1 = v_1v$. Then $\iota(v) = g$, $a_{k+1}v \in \mathcal{A}(\mathcal{B})$ and $x = (a_1u_1)(a_{k+1}u)(e_1u_{k+1})x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Hence

x, $(a_1u_1')(a_{k+1}v)(v_1w_1)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1b″. $s=0$.

We assert that $k \ge 2$. Indeed, assume to the contrary that $k=1$. Then $x_i = x_i'' = a_1a_2$, $x_j''=u_1$, $u_m''=u$, $3 \le |x_j| = 1+|x_j'|$, $3 \le |x_m| = 1+|x_m'|$, hence $|x_j'| \ge 2$, $|x_m'| \ge 2$, and therefore $|y_j'| = |y_m'| = 0$. Hence

$$\sum_{\nu=1}^r |y_\nu'| = |y_i'| \le |y_i| = 2 \quad \text{and} \quad \sum_{\nu=1}^r |x_\nu'| = |x_j'| + |x_m'| \ge 4,$$

a contradiction to $|x|_{\mathcal{B}} = |y|_{\mathcal{B}}$.

As $k \ge 2$, it follows that $u_2 \in \mathcal{A}(D_j)$, hence $u_2v_1 \in \mathcal{A}(\mathcal{B})$, and we choose $b_2 \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb_2$, whence $\iota(b_2) = \mathbf{0}$. Since $3 \le |x_m| = 1+|x_m'|$, we get $|x_m'| \ge 2$, and there exist v_1', $v_2' \in \mathcal{A}(D_m)$ such that $v_1'v_2' \mid x_m'$. Let $v \in \mathcal{A}(D_m)$ be such that $v_1'v_2' = v_1v$. Then $\iota(v) = g$ and $u_1v \in \mathcal{A}(\mathcal{B})$.

Assume first that $k \ge 2$, and set $x = (a_1u_1)(a_2u_2)v_1'v_2'x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \ge 1$. Then $u_2v_1 \in \mathcal{A}(\mathcal{B})$, and therefore x, $bb_2(u_1v)(u_2v_1)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.1c. $t=0$.

Observe that $|x_i''| = k \ge 2$ and $x = (a_1u_1)\cdot\ldots\cdot(a_ku_k)(e_1u_{k+1})\cdot\ldots\cdot(e_su_{k+s})\,\tilde{x}$. We may assume that there is no $v \in \mathcal{A}(\mathcal{B})$ such that $|v|=2$ and $v \mid \tilde{x}$. Indeed, suppose that $v \in \mathcal{A}(B)$ is such that $|v| = 2$ and $v \mid \tilde{x}$. Then $v = v'v''$, where v', $v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $\iota(v') = \iota(v'') = g$, and a_2v', $u_2v'' \in \mathcal{A}(\mathcal{B})$. We set $x = (a_1u_1)(a_2u_2)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, and $x' = (a_1u_1)(a_2v')(u_2v'')x^*$. Then it follows that $x' \in \mathsf{Z}(\mathcal{B})$, $(x,x') \in \sim_{\mathcal{B}}$ and $x \approx_{\mathrm{eq}} x'$. Hence $(x',y) \in \sim_{\mathcal{B}}$, $x' \approx_{\mathrm{eq}} y$ by Case 2.1b, and therefore $x \approx_{\mathrm{eq}} y$.

Next we prove that there is some $n \in [1,r] \setminus \{j\}$ such that $|x_n'| \ge 1$. Assume the contrary. Then $x = (a_1u_1)\cdot\ldots\cdot(a_ku_k)(e_1u_{k+1})\cdot\ldots\cdot(e_su_{k+s})x_j'$, $x_i = x_i'' = a_1\cdot\ldots\cdot a_k$, $x_j'' = u_1\cdot\ldots\cdot u_{k+s}$, and

$$e_1\cdot\ldots\cdot e_s = \prod_{\substack{\nu=0\\ \nu\ne i,j}}^r x_\nu\,.$$

Moreover, we obtain $|y_i''| = |y_i| - |y_i'| \le |x_i| - 2 = |x_i''| - 2 = k-2$ and $|y_j''| = |y_j| = |x_j'| + |x_j''| \ge 2+k+s$. We set $y_j'' = y_{j,1}\cdot\ldots\cdot y_{j,\mu}$, where $\mu = |y_j''|$, and, for all $\alpha \in [1,\mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1,\mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(B)$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. Since $|x_j'| \ge 1$, it follows that $u_{j,\alpha} \notin \mathcal{A}(D_j)$ for all $\alpha \in [1,\mu]$. For $\nu \in [0,r]\setminus\{j\}$, we set $N_\nu = |\{\alpha \in [1,\mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu\}|$, and we obtain

$$2+k+s \;\le\; \mu \;=\; \sum_{\substack{\nu=0\\ \nu\ne j}}^r N_\nu \;\le\; \sum_{\substack{\nu=0\\ \nu\ne j}}^r |y_\nu''| \;\le\; |y_i''| + \sum_{\substack{\nu=0\\ \nu\ne i,j}}^r |y_\nu| \;=\; |y_i''| + \sum_{\substack{\nu=0\\ \nu\ne i,j}}^r |x_\nu| \;\le\; k-2+s\,,$$

a contradiction.

Thus let now $n \in [1,r]\setminus\{j\}$ be such that $|x_n'| \ge 1$. Then $|x_n'| \ge 2$, $|y_n'| = 0$ and $|y_n''| \ge 2$. Let v_1, v_2, $w_1 \in \mathcal{A}(D_n)$ be such that $v_1v_2 \mid x_n'$, $w_1 \mid y_n''$, and choose some $x_2 \in \mathcal{A}(D_n)$ such that $v_1v_2 = w_1w_2$. Then $x = (a_1u_1)(a_2u_2)v_1v_2x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*| \ge 1$. Let $b_2 \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb_2$, whence $\iota(b_2) = \mathbf{0}$. Then x, $bb_2(u_1w_1)(u_2w_2)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.2. $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}} + 1$, and we are in the following special situation.

S1 There exist $a_1,\, a_2 \in \mathcal{A}(D_i)$ and $u_1,\, u_2 \in \mathcal{A}(F)\backslash\mathcal{A}(D_i)$ such that $a_1u_1,\, a_2u_2,\, u_1u_2 \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1)(a_2u_2) \mid_{\mathcal{B}} x$.

We set $x = (a_1u_1)(a_2u_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x|_{\mathcal{B}} \geq 1$, and we let $b' \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb'$, whence $\iota(b') = 0$. Then x, $bb'(u_1u_2)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.3. $|y|_{\mathcal{B}} = |x|_{\mathcal{B}} + 1$, and we are not in the special situation **S1**.
We set $x_i'' = a_1 \cdot \ldots \cdot a_k$, where $a_1, \ldots, a_k \in \mathcal{A}(D_i)$ and $k \geq 2$. For $\nu \in [1,k]$, let $u_\nu \in \mathcal{A}(F)$ be such that $a_\nu u_\nu \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1) \cdot \ldots \cdot (a_ku_k) \mid_{\mathcal{B}} x$. Since $|y_i'| \geq 1$ and we are not in the special situation **S1**, there exists some $j \in [1,r] \setminus \{i\}$ such that $u_1 \cdot \ldots \cdot u_k \mid x_j''$. Suppose that $x_j'' = u_1 \cdot \ldots \cdot u_{k+s}$, where $s \in \mathbb{N}_0$, and let $c_1, \ldots, c_s \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$ be such that $x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(c_1u_{k+1}) \cdot \ldots \cdot (c_su_{k+s})\tilde{x}$ for some $\tilde{x} \in \mathsf{Z}(\mathcal{B})$.
We may assume that there is no $v \in \mathcal{A}(\mathcal{B})$ such that $|v| = 2$ and $v \mid \tilde{x}$. Indeed, suppose that $v \in \mathcal{A}(B)$ is such that $|v| = 2$ and $v \mid \tilde{x}$. Then $v = v'v''$, where $v',\, v'' \in \mathcal{A}(F) \setminus (\mathcal{A}(D_i) \cup \mathcal{A}(D_j))$, $\iota(v') = \iota(v'') = g$, and $a_2v',\, u_2v'' \in \mathcal{A}(\mathcal{B})$. We set $x = (a_1u_1)(a_2u_2)(v'v'')x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$, and $x' = (a_1u_1)(a_2v')(u_2v'')x^*$. Then it follows that $x' \in \mathsf{Z}(\mathcal{B})$, $(x, x') \in \sim_{\mathcal{B}}$ and $x \approx_{\mathrm{eq}} x'$. Hence $(x', y) \in \sim_{\mathcal{B}}$, by Case 2.2, there is a monotone $\mathcal{R}$-chain concatenating x' and y, and therefore there is a monotone $\mathcal{R}$-chain concatenating x and y.
Hence x is of the form

$$x = (a_1u_1) \cdot \ldots \cdot (a_ku_k)(c_1u_{k+1}) \cdot \ldots \cdot (c_su_{k+s})x_1' \cdot \ldots \cdot x_r',$$

and we assert that there exists some $m \in [1,r] \setminus \{j\}$ such that $|x_m'| \geq 2$. Indeed, we assume to the contrary that $|x_m'| = 0$ for all $m \in [1,r] \setminus \{j\}$. Then we obtain $|x| = 2(k+s) + |x_j'|$, and since $|y|_{\mathcal{B}} = |x|_{\mathcal{B}} + 1$, it follows that

$$\sum_{\nu=0}^{r} |y_\nu'| = \sum_{\nu=0}^{r} |x_\nu'| + 2 = |x_j'| + 2.$$

If $|y_j'| \geq 1$, then we find $|x_j'| = 0$ and $|y_j'| \geq 2$, and therefore

$$4 \leq |y_j'| + |y_i'| \leq \sum_{\nu=0}^{r} |y_\nu'| = 2,$$

a contradiction. Hence it follows that $|y_j'| = 0$, and then $|y_j''| = |y_j| = |x_j| = k + s + |x_j'|$. Now we find

$$\sum_{\substack{\nu=0 \\ \nu \neq j}}^{r} |y_\nu''| \leq |y| - |y_j''| - |y_i'| \leq |x| - (k + s + |x_j'|) - 2 = k + s - 2 \leq |y_j''| - 2.$$

We set $y_j'' = y_{j_1} \cdot \ldots \cdot y_{j,\mu}$, where $\mu = |y_j''|$ and, for all $\alpha \in [1,\mu]$, $y_{j,\alpha} \in \mathcal{A}(D_j)$ and $\iota(y_{j,\alpha}) = g$. For $\alpha \in [1,\mu]$, let $u_{j,\alpha} \in \mathcal{A}(F)$ be such that $y_{j,\alpha}u_{j,\alpha} \in \mathcal{A}(\mathcal{B})$ and $y_{j,\alpha}u_{j,\alpha} \mid_{\mathcal{B}} y$. For $\nu \in [0,r]$, we set $N_\nu = \#\{\alpha \in [1,\mu] \mid y_{\nu,\alpha} \in \mathcal{A}(D_\nu)\}$, and we obtain

$$0 \leq \sum_{\substack{\nu=0 \\ \nu \neq j}}^{r} |y_\nu''| \leq |y_j''| - 2 = \sum_{\nu=0}^{r} N_\nu - 2,$$

and therefore $N_j \geq 2$. Hence, there exist $w_1,\, w_2 \in \mathcal{A}(D_j)$ such that $\iota(w_1) = \iota(w_2) = g$ and $w_1w_2 \in \mathcal{A}(\mathcal{B})$. On the other hand, $u_1u_2 \notin \mathcal{A}(\mathcal{B})$, since we are not in the special situation **S1**, and therefore $u_1u_2 = u_1'u_2'$, where $u_1',\, u_2' \in \mathcal{A}(D_j)$ and $\iota(u_1') = \iota(u_2') = \mathbf{0}$. Hence the existence of $w_1w_2 \in \mathcal{A}(\mathcal{B})$ contradicts **A1**.
Let now $m \in [1,r] \setminus \{j\}$ be such that $|x_m'| \geq 2$ and let $b' \in \mathcal{A}(D_i)$ be such that $a_1a_2 = bb'$.

By Reduction 1, we obtain $|y'_m| = 0$, hence $|y''_m| \geq 2$, and there exist v', $v'' \in \mathcal{A}(D_m)$ such that $v'v'' \mid x'_m$, and there exists some $u' \in \mathcal{A}(D_m)$ such that $u' \mid y''_m$. Let $u'' \in \mathcal{A}(D_m)$ be such that $v'v'' = u'u''$, whence $\iota(u'') = g$, and set $x = (a_1u_1)(a_2u_2)v'v''x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$. If $|x|_{\mathcal{B}} = 4$, then $x = (a_1u_1)(a_2u_2)v'v''$ and thus $y = y'_iy'_jy''_m$, where $|y'_i| = |y'_j| = |y''_m| = 2$, and thus there is a pure atom in m dividing y. Since v', $v'' \in \mathcal{A}(D_m)$ and $\iota(v') = \iota(v'') = \mathbf{0}$, this contradicts **A1**. Now we may assume $|x|_{\mathcal{B}} \geq 5$. Then we have $|x^*|_{\mathcal{B}} \geq 1$ and it follows that $u_1u', u_2u'' \in \mathcal{A}(\mathcal{B})$, and x, $bb'(u_1u')(u_2u'')x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 2.4. $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}} + 2$, and we are not in the special situation **S1**.
Let a_1, $a_2 \in \mathcal{A}(D_i)$ such that $a_1a_2 \mid x''_i$. Since $|y'_i| > 0$, there are u_1, $u_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that a_1u_1, $a_2u_2 \in \mathcal{A}(\mathcal{B})$ and $(a_1u_1)(a_2u_2) \mid_{\mathcal{B}} x$. We set $x = (a_1u_1)(a_2u_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*| \geq 1$, and $u_1u_2 = v_1v_2$ for some v_1, $v_2 \in \mathcal{A}(D_i)$ such that $\iota(v_1) = \iota(v_2) = \mathbf{0}$. Again we set $a_1a_2 = bb'$, where $b' \in \mathcal{A}(D_i)$ and $\iota(b') = \mathbf{0}$, and then x, $bb'v_1v_2x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y, since $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}} + 2 = |x^*| + 4$.
Reduction 2. By Case 2, we may now assume that $|y'_i| = 0$ for all $i \in [1, r]$, and since $|x|_{\mathcal{B}} \leq |y|_{\mathcal{B}}$, this implies that $|x'_i| = 0$ and therefore $|x''_i| \geq 2$ and $|y''_i| \geq 2$ for all $i \in [1, r]$. Since $x''_0 = x_0 = y_0 = y''_0$, we have $x_i = x''_i$ for all $i \in [0, r]$.
Case 3. $x_i = x''_i$, $y_i = y''_i$, and $|x_i| = |y_i| \geq 2$ for all $i \in [0, r]$.

Case 3a. There is some $i \in [0, r]$ such that

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |x_\nu| < |x_i| \quad \left[\text{and thus also } \sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |y_\nu| < |y_i|\right].$$

There exist a_1, a_2, b_1, $b_2 \in \mathcal{A}(D_i)$ such that $a_1a_2 \in \mathcal{A}(\mathcal{B})$, $b_1b_2 \in \mathcal{A}(\mathcal{B})$, $a_1a_2 \mid_{\mathcal{B}} x$, and $b_1b_2 \mid_{\mathcal{B}} y$. Let $b \in \mathcal{A}(D_i)$ be such that $a_1a_2 = b_1b$. Since $5 \leq |x|_{\mathcal{B}} \leq 2|x''_i| = 2|x_i|$, there exists some $a_w \in \mathcal{A}(D_i)$ such that $a_1a_2a_3 \mid x_i$. Let $c \in \mathcal{A}(F)$ be such that $a_3c \in \mathcal{A}(\mathcal{B})$ and $a_3c \mid_{\mathcal{B}} x$, and let $b_3 \in \mathcal{A}(D_i)$ be such that $ba_3 = b_2b_3$. If $x = (b_1b)(a_3c)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \geq 1$, then x, $(b_1b_2)(b_3c)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3b. For all $i \in [0, r]$, we have

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |x_\nu| \geq |x_i| \quad \left[\text{and thus also } \sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |y_\nu| \geq |y_i|\right].$$

We shall prove the following reduction step.

R1 We may assume that, for each $i \in [0, r]$, there is no pure atom in i dividing either x or y in $\mathcal{B}$.

Proof of **R1**. Let $\tilde{x} \in \mathsf{Z}(\mathcal{B})$ be such that $(x, \tilde{x}) \in \sim_{\mathcal{B}}$, $x \approx_{\text{eq}} \tilde{x}$, and the number of pure atoms dividing $\tilde{x}$ is minimal. Assume there is at least one pure atom in $i \in [0, r]$ dividing $\tilde{x}$, say $a_1a_2 \in \mathcal{A}(\mathcal{B})$ with a_1, $a_2 \in \mathcal{A}(D_i)$ and $a_1a_2 \mid_{\mathcal{B}} \tilde{x}$. Now we find

$$\sum_{\substack{\nu=0\\ \nu\neq i}}^{r} |\tilde{x}_\nu| \geq |\tilde{x}_i| - 2,$$

and thus there are c_1, $c_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ with $c_1c_2 \in \mathcal{A}(\mathcal{B})$ and $c_1c_2 \mid_{\mathcal{B}} \tilde{x}$. If $\tilde{x} = (a_1a_2)(c_1c_2)x^*$, where $x^* \in \mathcal{A}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \geq 1$, then we set $x' = (a_1c_1)(a_2c_2)x^*$. Now we find $(\tilde{x}, x') \in \sim_{\mathcal{B}}$ and $\tilde{x} \approx_{\text{eq}} x'$, and thus $x \approx_{\text{eq}} x'$. Since there is one pure atom less dividing x' than $\tilde{x}$, this is a contradiction.

The same argument applies to y. Therefore there exist $\tilde{x}$, $\tilde{y} \in \mathsf{Z}(\mathcal{B})$ both not divisable by any pure atom such that $(x, \tilde{x})$, $(y, \tilde{y}) \in \sim_{\mathcal{B}}$, $x \approx_{\text{eq}} \tilde{x}$, and $y \approx_{\text{eq}} \tilde{y}$. Hence it follows that $(\tilde{x}, \tilde{y}) \in \sim_{\mathcal{B}}$ and if $\tilde{x} \approx_{\text{eq}} \tilde{y}$, then $x \approx_{\text{eq}} y$. □

Next we prove the following reduction step.

R2 We may assume that, for each $i \in [0, r]$, $x_i = y_i$.

Proof of **R2**. Trivially, we have $x_0 = y_0$. Now let $i \in [1, r]$. We assert that there is some $\tilde{x} \in \mathsf{Z}(\mathcal{B})$ such that $(x, \tilde{x}) \in \sim_{\mathcal{B}}$, $x \approx_{\text{eq}} \tilde{x}$, and $z = \gcd(\tilde{x}_i, y_i)$ (in F) is maximal. Now assume that $\tilde{x}_i = z\tilde{z}$ and $y_i = z\tilde{z}'$, where $\tilde{z}$, $\tilde{z}' \in \mathsf{Z}(D_i)$ and $|\tilde{z}| = |\tilde{z}'| \geq 1$. If $|\tilde{z}| = |\tilde{z}'| = 1$, then there are some v, $v' \in \mathcal{A}(D_i)$ such that $\tilde{z} = v$ and $\tilde{z}' = v'$. Now we find

$$\pi_i(z)v = \pi_i(z\tilde{z}) = \pi_i(\tilde{x}_i) = \pi_i(y_i) = \pi_i(z\tilde{z}') = \pi_i(z)v',$$

and thus $v = v'$. But then $\gcd(\tilde{x}_i, y_i) = vz$, a contradiction. If $|\tilde{z}| = |\tilde{z}'| \geq 2$, then there are a_1, a_2, $b \in \mathcal{A}(D_i)$ with $a_1a_2 \mid \tilde{x}_i$ and $b \mid y_i$. By **R1**, there are c_1, $c_2 \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that a_1c_1, $a_2c_2 \in \mathcal{A}(\mathcal{B})$ and a_1c_1, $a_2c_2 \mid_{\mathcal{B}} x$. There is some $b' \in \mathcal{A}(D_i)$ such that $a_1a_2 = b'b$. If $\tilde{x} = (a_1c_1)(a_2c_2)x^*$, where $x^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} \geq 1$, then we set $\bar{x} = (bc_1)(b'c_2)x^*$ and find $(\tilde{x}, \bar{x}) \in \sim_{\mathcal{B}}$ and $\tilde{x} \approx_{\text{eq}} \bar{x}$, and thus $x \approx_{\text{eq}} \bar{x}$. Since $bz = b\gcd(\tilde{x}_i, y_i) \mid \gcd(\bar{x}_i, y_i)$, this is a contradiction. □

Now we fix—again arbitrarily—some $i \in [0, r]$ and choose $a \in \mathcal{A}(D_i)$ such that $a \mid x_i''$. Then $a \mid y_i''$, too. By **R1**, there are c, $d \in \mathcal{A}(F) \setminus \mathcal{A}(D_i)$ such that $ac \mid_{\mathcal{B}} x$ and $ad \mid_{\mathcal{B}} y$. Again by **R1**, there are e, $f \in \mathcal{A}(F)$ such that $de \mid_{\mathcal{B}} x$ and $cf \mid_{\mathcal{B}} y$.
Then x and y are of the following forms

$$x = (ac)(de)x^* \quad \text{and} \quad y = (ad)(cf)y^*,$$

where x^*, $y^* \in \mathsf{Z}(\mathcal{B})$ and $|x^*|_{\mathcal{B}} = |y^*|_{\mathcal{B}} \geq 1$.

Case 3.b$'$. $ce \in \mathcal{A}(\mathcal{B})$.

Then x, $(ad)(ce)x^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3.b$''$ $df \in \mathcal{A}(\mathcal{B})$.

Then x, $(ac)(df)y^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Case 3.b$'''$ We are neither in Case 3.b$'$ nor in Case 3.b$''$, and thus there are j_1, $j_2 \in [0, r] \setminus \{i\}$ with $j_1 \neq j_2$ such that c, $e \in \mathcal{A}(D_{j_1})$ and d, $f \in \mathcal{A}(D_{j_2})$. Then ae, af, $cd \in \mathcal{A}(\mathcal{B})$ and hence x, $(ae)(cd)x^*$, $(af)(cd)y^*$, y is a monotone $\mathcal{R}$-chain concatenating x and y.

Now it remains to prove the special case, where $|x|_{\mathcal{B}} = 3$. By the length formulas from the beginning of the proof we find that $|y|_{\mathcal{B}} \in \{5, 6\}$. If $|y|_{\mathcal{B}} = 5$, then the length formulas imply that that there is some $i \in [1, r]$ such that $|x_i'| = 1$ and $|y_i'| \geq 1$, and thus we are in the situation of Case 1.2. When we inspect the monotone $\mathcal{R}$-chain constructed there, we find that the same monotone $\mathcal{R}$-chain concatenating x and y exists in our situation. If $|y|_{\mathcal{B}} = 6$, then we find that $|x_i'| = 0$ and $|y_i''| = 0$ for all $i \in [1, r]$. Since $6 = |y|_{\mathcal{B}} \geq |x|_{\mathcal{B}} + 2 = 5$, we are either in Case 2.2 or in Case 2.4. Again we inspect the monotone $\mathcal{R}$-chains constructed there, and we find that the same monotone $\mathcal{R}$-chains concatenating x and y exist in our special situation.

Now it remains to show part 2. By Lemma 2.2.18.3, we have

$$\mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \le \sup\{|y| \mid (x,y) \in \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}}), \text{ there is no monotone } \mathcal{R}\text{-chain from } x \text{ to } y, \text{ and either } |x| = |y| \text{ or } |x|, |y| \in \mathsf{L}(\pi_{\mathcal{B}}(x)) \text{ are adjacent}\}.$$

By part 1, there is a monotone $\mathcal{R}$-chain concatenating x and y for all $(x,y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \ge 5$ and $|y|_{\mathcal{B}} \ge |x|_{\mathcal{B}}$. Thus it suffices to consider relations $(x,y) \in \sim_{\mathcal{B}}$ with $|x|_{\mathcal{B}} \le |y|_{\mathcal{B}} \le 4$. By definition we have $\{(x,y) \in \mathcal{A}(\sim_{\mathcal{B}}) \mid |x|_{\mathcal{B}} \le |y|_{\mathcal{B}}\} \subset \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}})$. Since the shortest possible atom $(x,y) \in \mathcal{A}(\sim_{\mathcal{B}}) \setminus \mathcal{A}(\sim_{\mathcal{B},\mathrm{mon}})$ satisfies $|x|_{\mathcal{B}} > |y|_{\mathcal{B}} \ge 2$, we find $|xy|_{\mathcal{B}} \ge 5$. Hence, we may restrict on elements of $\mathcal{A}(\sim_{\mathcal{B}})$ and the assertion follows. $\square$

Using Lemma 3.1.9.4, Lemma 3.1.11, and Lemma 3.1.12 above, we can now calculate the catenary degree and the minimum distance (with $\#G = 2$), and in a slightly more special but still interesting situation, we can compute the elasticity, the monotone catenary degree and the tame degree.

Proposition 3.1.13. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}_0$, $s \in \mathbb{N}_0$, $r + s \ge 1$, and let $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ be reduced half-factorial but not factorial monoids of type $(1, k_i)$ for $i \in [1, r+s]$ with $k_1 = \ldots = k_r = 1$ and $k_{r+1} = \ldots = k_s = 2$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_{r+s}$. Let $H \subset D$ be a saturated submonoid, let $G = \mathsf{q}(D/H)$ be its class group with $\#G = 2$, say $G = \{\mathbf{0}, g\}$, let each class in G contain some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_{r+s} \to G$ by $\iota(t) = [t]_{D/H}$. Furthermore, set $I = \{i \in [1, r+s] \mid (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2 \ne \emptyset\}$ and $J = \{i \in [r+1, r+s] \mid \mathsf{c}(D_i) = 3\}$. Then*

1. *If $I = J = \emptyset$, then H is half-factorial and $\mathsf{c}(H) = 2$.*
2. *If $I = \emptyset$ and $J \ne \emptyset$, then $\mathsf{c}(H) \in \{2, 3\}$, and $\triangle(H) \subset \{1\}$.*
3. *If $\#I = 1$, then $\rho(H) \ge \frac{3}{2}$, $\mathsf{c}(H) = 3$, and $\triangle(H) = \{1\}$.*
4. *If $\#I \ge 2$, then $\rho(H) = 2$, $\mathsf{c}(H) = 4$, and $\triangle(H) = \{1, 2\}$.*
5. *If $s = 0$, then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H)$. Additionally, if $\#I = 1$, then $\rho(H) = \frac{3}{2}$.*
6. *If $s = 0$ and $\iota(p_i) = \mathbf{0}$ for all $i \in [1, r]$, then H is half-factorial if and only if $\mathsf{t}(H) = 2$.*

In particular, $\min \triangle(H) \le 1$ always holds.

Proof. We set $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_{r+s} \mid \mathbf{0} \nmid S\}$. By Lemma 1.2.16, H and D are reduced BF-monoids, and $H \subset D$ is a faithfully saturated submonoid. By Lemma 1.2.16.4, we obtain $\triangle(H) = \triangle(\mathcal{B})$, $\rho(H) = \rho(\mathcal{B})$, $\mathsf{c}(H) = \mathsf{c}(\mathcal{B})$, and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B})$. Lemma 3.1.9.1 implies $\mathsf{c}(D) \le 3$, and, by Lemma 3.1.9.4, we obtain $\mathsf{c}(\mathcal{B}) = \mathsf{c}(H) \le 4$.
By Proposition 2.1.9.1, we obtain $\mathsf{c}(\mathcal{B}) \le \sup\{|y|_{\mathcal{B}} \mid (x,y) \in \mathcal{A}(\sim_{\mathcal{B}})$, and since $\mathsf{c}(\mathcal{B}) \le 4$, it follows that

$$\mathsf{c}(\mathcal{B}) \le \max\{|y|_{\mathcal{B}} \mid (x,y) \in \mathcal{A}(\sim_{\mathcal{B}}),\ |x|_{\mathcal{B}} \le |y|_{\mathcal{B}} \le 4\};$$

indeed we can replace the supremum with a maximum since we have a bounded set of integers on the right hand side.

If $(x,y) \in \mathcal{A}(\sim_{\mathcal{B}})$, then $(x,y) = (u_1 \cdot \ldots \cdot u_k, v_1 \cdot \ldots \cdot v_l)$, where $k = |x|_{\mathcal{B}}$, $l = |y|_{\mathcal{B}}$, and u_i, $v_j \in \mathcal{A}(\mathcal{B})$ for all $i \in [1,k]$ and $j \in [1,l]$. In this case, we call the atom (x,y) of type

(k,l) and describe it by the defining relation $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$ in $\mathcal{B}$. Now the equation from above reads the following

$$\mathsf{c}(\mathcal{B}) = \max\{|x|_{\mathcal{B}} \mid (x,y) \in \mathcal{A}(\sim_{\mathcal{B}}) \text{ is of type } (k,l), \text{ where } 2 \le k \le l \le 4\}.$$

Hence we proceed with a list of defining relations for all atoms of type (k,l), where $2 \le k \le l \le 4$. An atom will be called of character $\mathcal{C} \in [1,15]$ if it is defined by the relation (3.1.$\mathcal{C}$) in the list below.
Let i, $j \in [1, r+s]$, $i \ne j$. Then

$$g^2(p_ip_j\varepsilon_i\varepsilon_j) = (p_i\varepsilon_i g)(p_j\varepsilon_j g) \tag{3.1.1}$$

describes an atom of type $(2,2)$ if and only if $\varepsilon_i \in \mathcal{U}_1(D_i)$, $\varepsilon_j \in \mathcal{U}_1(D_j)$, and $\iota(p_i\varepsilon_i) = \iota(p_j\varepsilon_j) = g$;

$$(p_ip_j\varepsilon_i^{(1)}\varepsilon_j^{(1)})(p_ip_j\varepsilon_i^{(2)}\varepsilon_j^{(2)}) = (p_i^2\varepsilon_i^{(1)}\varepsilon_i^{(2)})(p_j^2\varepsilon_j^{(1)}\varepsilon_j^{(2)}) \tag{3.1.2}$$

describes an atom of type $(2,2)$ if and only if $\iota(p_i\varepsilon_i^{(1)}) = \iota(p_i\varepsilon_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\varepsilon_j^{(2)}) = g$, $\varepsilon_i^{(1)}\varepsilon_i^{(2)} \notin \mathcal{U}_1(D_i)^2_{\iota(p_i)}$, and $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$;

$$g^2(p_i^2\varepsilon_1\varepsilon_2) = (p_i\varepsilon_1 g)(p_i\varepsilon_2 g) \tag{3.1.3}$$

describes an atom of type $(2,2)$ if and only if either ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$, $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_g)^2$, and $\iota(p_i) = g$ or ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_g$, $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2$, and $\iota(p_i) = \mathbf{0}$;

$$(p_i\varepsilon_1)(p_i\varepsilon_2) = (p_i\eta_1)(p_i\eta_2) \tag{3.1.4}$$

describes an atom of type $(2,2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\eta_1) = \iota(\eta_2)$, and $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$;

$$(p_i\varepsilon_1 g)(p_i\varepsilon_2 g) = (p_i\eta_1 g)(p_i\eta_2 g) \tag{3.1.5}$$

describes an atom of type $(2,2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\eta_1) = \iota(p_i\eta_2) = g$, and $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$;

$$(p_i\varepsilon_1)(p_i\varepsilon_2 g) = (p_i\eta_1)(p_i\eta_2 g) \tag{3.1.6}$$

describes an atom of type $(2,2)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\iota(p_i) = \iota(\varepsilon_1) = \iota(\eta_1)$, $\iota(p_i\varepsilon_2) = \iota(p_i\eta_2) = g$, and $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$; and

$$(p_i\varepsilon_1 g)(p_i\varepsilon_2 g) = (p_i\eta_1)(p_i\eta_2)g^2\,, \tag{3.1.7}$$

describes an atom of type $(2,3)$ if and only if ε_1, ε_2, η_1, $\eta_2 \in \mathcal{U}_1(D_i)$, $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = g$, and $\iota(p_i\eta_1) = \iota(p_i\eta_2) = \mathbf{0}$. If these conditions are fulfilled, then $\varepsilon_1\varepsilon_2 \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$ and therefore $i \in I$. Conversely, if $i \in I$, then $\mathcal{U}_1(D_1)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g \ne \emptyset$. If $\iota(p_i) = g$, let ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$ and η_1, $\eta_2 \in \mathcal{U}_1(D_i)_g$ be such that $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$. If $\iota(p_i) = \mathbf{0}$, let ε_1, $\varepsilon_2 \in \mathcal{U}_1(D_i)_g$ and η_1, $\eta_2 \in \mathcal{U}_1(D_i)_{\mathbf{0}}$ be such that $\varepsilon_1\varepsilon_2 = \eta_1\eta_2$. In any case, (3.1.7) holds.
Now let $i \in I$, $j \in [1, r+s]$, and $i \ne j$. Then

$$(p_ip_j\varepsilon_i^{(1)}\varepsilon_j^{(1)})(p_ip_j\varepsilon_i^{(2)}\varepsilon_j^{(2)}) = (p_i\eta_i^{(1)})(p_i\eta_i^{(2)})(p_j^2\varepsilon_j^{(1)}\varepsilon_j^{(2)}) \tag{3.1.8}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_i^{(1)}, \varepsilon_i^{(2)}, \eta_i^{(1)}, \eta_i^{(2)} \in \mathcal{U}_1(D_i)$, $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)$, $\varepsilon_i^{(1)}\varepsilon_i^{(2)} = \eta_i^{(1)}\eta_i^{(2)}$, $\iota(p_i\varepsilon_i^{(1)}) = \iota(p_i\varepsilon_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\varepsilon_j^{(2)}) = g$, $\iota(p_i\eta_i^{(1)}) = \iota(p_i\eta_i^{(2)}) = \mathbf{0}$, and $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$. If these conditions are fulfilled, then $\varepsilon_i^{(1)}\varepsilon_i^{(2)} \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$

and therefore $i \in I$. However, if $i \in I$, then a relation (3.1.8) need not hold, since we cannot guarantee that there exist $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)$ such that $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \notin \mathcal{U}_1(D_j)^2_{\iota(p_j)}$.
Now let i, $j \in I$ and $i \neq j$. Then

$$(p_i p_j \varepsilon_i^{(1)}\varepsilon_j^{(1)})(p_i p_j \varepsilon_i^{(2)}\varepsilon_j^{(2)}) = (p_i\eta_i^{(1)})(p_i\eta_i^{(2)})(p_j\eta_j^{(1)})(p_j\eta_j^{(2)}) \tag{3.1.9}$$

describes an atom of type $(2,4)$ if and only if $\varepsilon_i^{(1)}$, $\varepsilon_i^{(2)}$, $\eta_i^{(1)}$, $\eta_i^{(2)} \in \mathcal{U}_1(D_i)$, $\varepsilon_j^{(1)}$, $\varepsilon_j^{(2)}$, $\eta_j^{(1)}$, $\eta_j^{(2)} \in \mathcal{U}_1(D_j)$, $\varepsilon_i^{(1)}\varepsilon_i^{(2)} = \eta_i^{(1)}\eta_i^{(2)}$, $\varepsilon_j^{(1)}\varepsilon_j^{(2)} = \eta_j^{(1)}\eta_j^{(2)}$, $\iota(p_i\varepsilon_i^{(1)}) = \iota(p_i\varepsilon_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\varepsilon_j^{(2)}) = g$, and $\iota(p_i\eta_i^{(1)}) = \iota(p_i\eta_i^{(2)}) = \iota(p_j\varepsilon_j^{(1)}) = \iota(p_j\eta_j^{(2)}) = \mathbf{0}$. If these conditions are fulfilled, then $\varepsilon_i^{(1)}\varepsilon_i^{(2)} \in \mathcal{U}_1(D_i)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_i)^2_g$ and $\varepsilon_j^{(1)}\varepsilon_j^{(2)} \in \mathcal{U}_1(D_j)^2_{\mathbf{0}} \cap \mathcal{U}_1(D_j)^2_g$, and therefore i, $j \in I$. Conversely, if i, $j \in I$, then a relation (3.1.9) holds (see the arguments for (3.1.7)).
Let $i \in J$, ε_1, ε_2, ε_3, η_1, η_2, $\eta_3 \in \mathcal{U}_1(D_i)$, and $\mathsf{c}_{D_i}((p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3), (p_i\eta_1)(p_i\eta_2)(p_i\eta_3)) = 3$. Then

$$(p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3) \tag{3.1.10}$$

describes an atom of type $(3,3)$ if and only if $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\varepsilon_3) = \iota(\eta_1) = \iota(\eta_2) = \iota(\eta_3)$;

$$(p_i\varepsilon_1)(p_i\varepsilon_2)(p_i\varepsilon_3 g) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3 g) \tag{3.1.11}$$

describes an atom of type $(3,3)$ if and only if $\iota(p_i) = \iota(\varepsilon_1) = \iota(\varepsilon_2) = \iota(\eta_1) = \iota(\eta_2)$ and $\iota(p_i\varepsilon_3) = \iota(p_i\eta_3) = g$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3) \tag{3.1.12}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = g$, and $\iota(p_i) = \iota(\varepsilon_1) = \iota(\eta_1) = \iota(\eta_2) = \iota(\eta_3)$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3 g) = (p_i\eta_1)(p_i\eta_2)(p_i\eta_3 g) \tag{3.1.13}$$

describes an atom of type $(2,3)$ if and only if $\varepsilon_1\varepsilon_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\varepsilon_3) = \iota(p_i\eta_3) = g$, and $\iota(p_i) = \iota(\eta_1) = \iota(\eta_2)$;

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3) = (p_i^2\eta_1\eta_2)(p_i\eta_3)\,, \tag{3.1.14}$$

describes an atom of type $(2,2)$ if and only if $\varepsilon_1\varepsilon_2$, $\eta_1\eta_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\eta_1) = \iota(p_i\eta_2) = g$, and $\iota(p_i) = \iota(\varepsilon_3) = \iota(\eta_3)$; and

$$(p_i^2\varepsilon_1\varepsilon_2)(p_i\varepsilon_3 g) = (p_i^2\eta_1\eta_2)(p_i\eta_3 g) \tag{3.1.15}$$

describes an atom of type $(2,2)$ if and only if $\varepsilon_1\varepsilon_2$, $\eta_1\eta_2 \notin (\mathcal{U}_1(D_i)_{\mathbf{0}})^2 \cap (\mathcal{U}_1(D_i)_g)^2$, and $\iota(p_i\varepsilon_1) = \iota(p_i\varepsilon_2) = \iota(p_i\varepsilon_3) = \iota(p_i\eta_1) = \iota(p_i\eta_2) = \iota(p_i\eta_3) = g$.
Now we can do the actual proof.

1. If $I = J = \emptyset$, then only atoms of characters $[1,6]$ exist, and they all are of type $(2,2)$. Hence, we obtain $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}) = 2$, and thus H is half-factorial.
2. If $I = \emptyset$ and $J \neq \emptyset$, then there are atoms of characters $[1,6] \cup [10,15]$, and they are of types $(2,2)$, $(2,3)$, and $(3,3)$. Hence, it follows that $\mathsf{c}(H) \in \{2,3\}$ and $\triangle(H) \subset \{1\}$.
3. If $\#I = 1$, then atoms of characters $[1,7]$ exist and atoms of characters $\{8\} \cup [10,15]$ might exist. The atoms of characters $[1,7]$ are of types $(2,2)$ and $(2,3)$ and the atoms of characters $\{8\} \cup [10,15]$ are of types $(2,3)$, $(3,3)$, and $(2,2)$. Thus we have $\rho(H) \geq \frac{3}{2}$ and $\mathsf{c}(H) = 3$, and therefore $\triangle(H) = \{1\}$ by Lemma 1.2.9.4.

4. If $\#I \geq 2$, then atoms of characters $[1,7] \cup \{9\}$ exist and possibly also atoms of characters $\{8\} \cup [10,15]$ exist, and they are of types $(2,2)$, $(2,3)$, and $(2,4)$. Thus we find $\mathsf{c}(H) = 4$, $\{1,2\} \subset \triangle(H)$, and $\rho(H) \geq 2$. Since $\rho(H) \leq 2$ by Lemma 3.1.9.4, we obtain the equality $\rho(H) = 2$ and, by Lemma 1.2.9.4, we find $\triangle(H) = \{1,2\}$.
5. Let $s = 0$. If $I = \emptyset$, then H is half-factorial by part 1, and thus $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H)$ by Lemma 2.2.19.1.

   If $\#I = 1$, then atoms of characters $[1,7]$ exist and atoms of character 8 might exist. The atoms of characters $[1,7]$ are of types $(2,2)$ and $(2,3)$ and the atoms of character 8 are also of type $(2,3)$. By Lemma 3.1.12.2, we have $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \leq 3$. By part 3, we find $3 = \mathsf{c}(H) \leq \mathsf{c}_{\mathrm{mon}}(H)$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = 3$.
   It remains to show that $\rho(H) = \rho(\mathcal{B}) = \frac{3}{2}$. By part 3, we have $\rho(H) \geq \frac{3}{2}$. Thus it suffices to show that $\rho(H) \leq \frac{3}{2}$. Now let $(x,y) \in \sim_{\mathcal{B}}$ with $|y|_{\mathcal{B}} \geq |x|_{\mathcal{B}}$. Then there is a monotone 3-chain concatening x and y, say $x = z_0, z_1, \ldots, z_n = y$ with $z_1, \ldots, z_n \in \mathsf{Z}(\pi_{\mathcal{B}}(x))$ and $n \in \mathbb{N}$. Whenever $|z_{i-1}|_{\mathcal{B}} < |z_i|_{\mathcal{B}}$ for some $i \in [1,n]$, then $\mathsf{d}(z_{i-1}, z_i) = 3$ and there is an atom $(z'_{i-1}, z'_i) \in \mathcal{A}(\sim_H)$ of charachter 7 or 8 such that $z_{i-1} = d_i z'_{i-1}$ and $z_i = d_i z'_i$, where $d_i = \gcd(z_{i-1}, z_i)$. Since atoms of both characters, replace two very special atoms in $\mathcal{A}(\mathcal{B})$ (on the left side) by three different atoms (on the right side) and there is no atom of character $x \in [1,6]$, which generates the first special atoms, there are at most $\frac{1}{2}|x|_{\mathcal{B}}$ such steps, and thus $|y|_{\mathcal{B}} \leq \frac{3}{2}|x|_{\mathcal{B}}$, what proves $\rho(H) \leq \frac{3}{2}$.

   If $\#I \geq 2$, then atoms of characters $[1,7] \cup \{9\}$ exist and possibly also atoms of character 8 exist. The atoms of characters $x \in [1,7] \cup \{9\}$ are of types $(2,2)$, $(2,3)$, and $(2,4)$ and the atoms of character 8 are of type $(2,3)$. By Lemma 3.1.12.2, we have $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \leq 4$ and, by part 4, we obtain $4 = \mathsf{c}(H) \leq \mathsf{c}_{\mathrm{mon}}(H)$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = 4$.

In order to finish the proof, we need an additional Lemma.

LEMMA 3.1.14. *Let D be a monoid, $P \subset D$ be a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset \widehat{D_i} = [p_i] \times \widehat{D_i}^{\times}$ reduced half-factorial monoids of type $(1,1)$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, let $G = \mathsf{q}(D/H)$ be its class group with $\#G = 2$, say $G = \{\mathbf{0}, g\}$, let each class in G contain some $p \in P$, and define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$. Furthermore, let $\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$ be the $(D_1 \times \ldots \times D_r)$-block monoid defined by ι over G and let $\mathcal{B}$ be half-factorial but not factorial.*
Then $\mathsf{t}(H) = \mathsf{t}(\mathcal{B}) = 2$.

PROOF. Throughout the proof, we write $\mathcal{B} = \{S \in \mathcal{B}(G, D_1 \times \ldots \times D_{r+s}, \iota) \mid \mathbf{0} \nmid S\}$ as in Lemma 1.2.16.4. By Proposition 3.1.13.1-4, we find that $\{i \in [1,r] \mid (\mathcal{U}_1(D_i))_{\mathbf{0}} \cap (\mathcal{U}_1(D_i))_g \neq \emptyset\} = \emptyset$, and thus $\iota(\widehat{D_i}^{\times}) = \{\mathbf{0}\}$ for all $i \in [1,r]$. Now let $h \in H$, $z \in \mathsf{Z}(h)$, and $a \in \mathcal{A}(H)$ be such that $a \mid h$. Then we prove that $\mathsf{d}(z,z') \leq 2$ for some $z' \in \mathsf{Z}(h) \cap a\mathsf{Z}(H)$. We may assume that $a \nmid z$. We find that z is of the following form:

$$z = q_1 \cdot \ldots \cdot q_k (q_1^{(1)} q_1^{(2)}) \cdot \ldots \cdot (q_l^{(1)} q_l^{(2)}) t_1 \cdot \ldots \cdot t_m,$$

where $q_1, \ldots, q_k$, $q_1^{(1)}$, $q_1^{(2)}, \ldots, q_l^{(1)}$, $q_l^{(2)} \in P$, $[q_1]_{D/H} = \ldots = [q_k]_{D/H} = \mathbf{0}$, $[q_1^{(1)}]_{D/H} = [q_1^{(2)}]_{D/H} = \ldots = [q_l^{(1)}]_{D/H} = [q_l^{(2)}]_{D/H} = g$, and $t_1, \ldots, t_m \in \mathcal{A}(D_1 \times \ldots \times D_r)$. Now we have the following three possibilities for a.

$$\begin{aligned} a &= \bar{q} && \text{with } \bar{q} \in P,\ [\bar{q}]_{D/H} = \mathbf{0}, \text{ or} \\ a &= \bar{q}^{(1)}\bar{q}^{(2)} && \text{with } \bar{q}^{(1)},\ \bar{q}^{(2)} \in P,\ [\bar{q}^{(1)}]_{D/H} = [\bar{q}^{(2)}]_{D/H} = g, \text{ or} \\ a &= u && \text{with } u \in \mathcal{A}(D_i) \text{ for some } i \in [1, r]. \end{aligned}$$

We proceed case by case. Let $a = \bar{q}$, where $\bar{q} \in P$ and $[\bar{q}]_{D/H} = \mathbf{0}$. Since $q_1, \ldots, q_k$, $q_1^{(1)}$, $q_1^{(2)}, \ldots, q_l^{(1)}$, $q_l^{(2)} \in P$ are prime in D and since $[\bar{q}]_{D/H} = \mathbf{0}$, we find $\bar{q} \in \{q_1, \ldots, q_k\}$. Thus $a = \bar{q} \mid z$, a contradiction. Let $a = \bar{q}^{(1)}\bar{q}^{(2)}$, where $\bar{q}^{(1)}$, $\bar{q}^{(2)} \in P$ and $[\bar{q}^{(1)}]_{D/H} = [\bar{q}^{(2)}]_{D/H} = g$. By the same arguments as before, we find $\bar{q}^{(1)}$, $\bar{q}^{(2)} \in \{q_1^{(1)}, q_1^{(2)}, \ldots, q_l^{(1)}, q_l^{(2)}\}$. Since $a \nmid z$, there is no $i \in [1, l]$ such that without loss of generality $\bar{q}^{(j)} = \bar{q}_i^{(j)}$ for $j = 1, 2$. Thus there are i, $j \in [1, l]$ with $i \neq j$ such that without loss of generality $\bar{q}^{(1)} = q_i^{(1)}$ and $\bar{q}^{(2)} = q_j^{(2)}$. Now we find the factorization $z' \in \mathsf{Z}(h)$,

$$z' = q_1 \cdot \ldots \cdot q_k (q_i^{(1)} q_j^{(2)})(q_j^{(1)} q_i^{(2)}) \prod_{s=1 s \neq i,j}^{s=l} (q_s^{(1)} q_s^{(2)}) t_1 \cdot \ldots \cdot t_m,$$

such that $\mathsf{d}(z, z') = 2$ and $a \mid z'$. Lastly, we consider the case $a = u$ with $u \in \mathcal{A}(D_i)$ for some $i \in [1, r]$. Then there are $u_1, \ldots, u_{\bar{m}} \in \mathcal{A}(D_1 \times \ldots \times D_r)$ such that

$$t_1 \cdot \ldots \cdot t_m = u u_1 \cdot \ldots \cdot u_{\bar{m}} \quad \text{and } \mathsf{d}_{D_1 \times \ldots \times D_r}(t_1 \cdot \ldots \cdot t_m, u u_1 \cdot \ldots \cdot u_{\bar{m}}) \leq 2.$$

Now we find a factorization $z' \in \mathsf{Z}(h)$ by setting

$$z' = q_1 \cdot \ldots \cdot q_k (q_1^{(1)} q_1^{(2)}) \cdot \ldots \cdot (q_l^{(1)} q_l^{(2)}) u u_1 \cdot \ldots \cdot u_{\bar{m}},$$

and $\mathsf{d}(z, z') \leq 2$ follows. □

6. Let $s = 0$ and $\iota(p_i) = \mathbf{0}$ for all $i \in [1, r]$. If H is not half-factorial, then $\mathsf{c}(H) \geq 3$ and therefore $\mathsf{t}(H) \geq 3$. Otherwise, if H is half-factorial, then $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}) = 2$, and therefore $I = \emptyset$ by points 1-4. Thus $\iota(u) = \iota(p_i) = \mathbf{0}$ for all $u \in \mathcal{A}(D_i)$ and $i \in [1, r]$, and any $a \in \mathcal{A}(\mathcal{B})$ is either of the form $a = g^2$ or $a = u$ with $u \in \mathcal{A}(D_i)$ for some $i \in [1, r]$. Since, by Lemma 3.1.3.2, $\mathsf{t}(D_i) = 2$ for all $i \in [1, r]$, we have $\mathsf{t}(\mathcal{B}) = 2$. Now the assertion follows by Lemma 3.1.14. □

The following example shows that the very special structure of D in the hypothesis of Lemma 3.1.14—in terms of Example 3.1.15 the structure T—is definitely necessary for the assertion of Lemma 3.1.14 to hold.

Example 3.1.15. Let P be a set of prime elements and T be an atomic monoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated submonoid with class group $D/H = C_2$ such that each class in C_2 contains some $p \in P$. Let $\iota : T \to C_2$, $t \mapsto [t]_{D/H}$ be a homomorphism and $\mathcal{B}(C_2, T, \iota)$ the T-block monoid over C_2 defined by ι. Furthermore let $\mathsf{t}(\mathcal{B}(C_2, T, \iota)) = 2$.
This situation does not imply $\mathsf{t}(H) = 2$.

Proof. We write $C_2 = \{\mathbf{0}, g\}$ and we set $\mathcal{B}(C_2, T, \iota)$ and denote by $\beta : H \to \mathcal{B}$ the block homomorphism of H and by $\bar{\beta} : \mathsf{Z}(H) \to \mathsf{Z}(\mathcal{B})$ the canonical extension of the block homomorphism.

By definition, it is sufficient to prove $\mathsf{t}(a,v) \geq 3$ for some $a \in H$ and some $v \in \mathcal{A}(H)$. Let $a \in H$ and $v \in \mathcal{A}(H)$.
We have the following for types of atoms of H which are not prime:

$$\begin{aligned}
v &= p_1p_2 && \text{with } p_1,\, p_2 \in P \text{ and } [p_1]_{D/H} = [p_2]_{D/H} = g\\
v &= pt && \text{with } p \in P,\, t \in T \text{ and } [p]_{D/H} = [t]_{D/H} = g\\
v &= t_1t_2 && \text{with } t_1,\, t_2 \in \mathcal{A}(T) \text{ and } [t_1]_{D/H} = [t_2]_{D/H} = g\\
v &= t && \text{with } t \in \mathcal{A}(T) \text{ and } [t]_{D/H} = g
\end{aligned}$$

Let $z \in \mathsf{Z}_H(a)$. Without loss of generality, we may assume that no prime elements divide a. Then z is of the following form:

$$z = (p_1p_2) \cdot \ldots \cdot (p_{l-1}p_l)(p_{l+1}s_1) \cdot \ldots \cdot (p_{l+m}s_m)(t_1t_2) \cdot \ldots \cdot (t_{n-1}t_n)u_1 \cdot \ldots \cdot u_o.$$

Let $v = q_1q_2$ be of the first type. Since all $p \in P$ are prime in D, we find $i,\, j \in [1, l+m]$ such that $p_i = q_1$ and $p_j = q_2$. Assume $i = l+1$ and $j = l+2$. Then we find

$$z' = (p_{l+1}p_{l+2})(p_1s_1)(p_2s_2)(p_1p_2)^{-1}(p_{l+1}s_1)^{-1}(p_{l+2}s_2)^{-1}z.$$

Thus $\mathsf{d}(z, z') = 3$. If we apply $\bar{\beta}$ to z', we find

$$\bar{\beta}(z') = g^2(gs_1)(gs_2)(g^2)^{-1}(gs_1)^{-1}(gs_2)^{-1}\bar{\beta}(z) = \bar{\beta}(z),$$

and thus $\mathsf{d}(\bar{\beta}(z), \bar{\beta}(z')) = 0$. □

Corollary 3.1.16. *Let D be an atomic monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and let $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ be reduced half-factorial monoids of type $(1,1)$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ its class group, and let each class in G contain some $p \in P$. Then the following are equivalent:*

- $\mathsf{c}_{\mathrm{mon}}(H) \leq 2$.
- $\mathsf{c}(H) \leq 2$.
- *H is half-factorial.*

If, additionally, $[p_i]_{D/H} = \mathbf{0}_{D/H}$ for all $i \in [1, r]$—in particular, this is true if $\#G = 1$—then the following is also equivalent:

- $\mathsf{t}(H) \leq 2$.

Proof. By Lemma 3.1.9.3, $\#G \geq 3$ implies $\mathsf{c}(H) \geq 3$ and thus that H is never half-factorial. Thus we have $\#G \leq 2$. If $\#G = 2$, then the assertion follows by Proposition 3.1.13. If $\#G = 1$, the assertion follows by Lemma 3.1.9.2 and Lemma 3.1.9.1. □

Lemma 3.1.17. *Let $\mathcal{O}$ be a locally half-factorial order in an algebraic number field. Then there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$, $\mathcal{I}^*(\mathcal{O}) \cong D$, $\mathcal{O}^\bullet_{\mathrm{red}} \subset D$ is a saturated submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathsf{q}(D/\mathcal{O}^\bullet_{\mathrm{red}})$ is its class group, and each class contains some $p \in P$. If, additionally, all localizations of $\mathcal{O}$ are finitely primary monoids of exponent 1, then $k_i = 1$ for all $i \in [1, r]$.*

Proof. Let $\mathcal{O}$ be an order in an algebraic number field and let $\mathcal{I}^*(\mathcal{O})$ be half-factorial. We set $\overline{\mathcal{O}}$ for the integral closure of $\mathcal{O}$ and set $\mathfrak{f} = (\mathcal{O} : \overline{\mathcal{O}})$, $\mathcal{P} = \{p \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \not\supset \mathfrak{f}\}$, $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset \mathfrak{f}\}$, and

$$T = \prod_{\mathfrak{p} \in \mathcal{P}^*} (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}.$$

By [**14**, Theorem 3.7.1], we find that $\mathcal{P}^*$ is finite, $\mathcal{O}_{red}^\bullet \subset \mathcal{F}(\mathcal{P}) \times T$ is a saturated and cofinal submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathcal{C}_\mathsf{v}(\mathcal{O}) = (\mathcal{F}(\mathcal{P}) \times T)/\mathcal{O}_{red}^\bullet$, and, for all $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$, $\mathcal{O}_\mathfrak{p}^\bullet$ is a finitely primary monoid of rank $s_\mathfrak{p}$, where $s_\mathfrak{p}$ is the number of prime ideals $\overline{\mathfrak{p}} \in \mathfrak{X}(\overline{\mathcal{O}})$ such that $\overline{\mathfrak{p}} \cap \mathcal{O} = \mathfrak{p}$. For $\mathfrak{p} \in \mathcal{P}^*$, the local domain $\mathcal{O}_\mathfrak{p}$ is not integrally closed, hence not factorial, and therefore the monoid $(\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ is not factorial, too. Since $\mathcal{I}^*(\mathcal{O}) \cong \prod_{\mathfrak{p} \in \mathfrak{X}(\mathcal{O})} (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ by [**14**, Theorem 3.7.1], we find, for all $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$, that $\mathcal{O}_\mathfrak{p}$ is half-factorial, and thus, by the additional statement in Lemma 3.1.3.1, $\mathcal{O}_\mathfrak{p}^\bullet$ is a half-factorial monoid of type $(1, k_\mathfrak{p})$, where $k_\mathfrak{p}$ is the rank of $\mathcal{O}_\mathfrak{p}^\bullet$. By [**18**, Corollary 3.5], we find $k_\mathfrak{p} \leq 2$. Now we set $D_i = (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$ for some $\mathfrak{p} \in \mathcal{P}^*$ such that $T = D_1 \times \ldots \times D_r$ and we set $P = \mathcal{P}$. By [**14**, Corollary 2.11.16], every class contains infinitely many primes $p \in P$. Since, by the above, k_i is the exponent of D_i for all $i \in [1, r]$, the additional statement is obvious. $\square$

In the hypotheses of the following theorem we restrict to locally half-factorial orders in algebraic number fields. This seems reasonable since there are no known half-factorial orders, which are not locally half-factorial, and—see Corollary 3.1.24—this assumption is satisfied for orders in quadratic number fields.

Theorem 3.1.18. *Let $\mathcal{O}$ be a non-principal locally half-factorial order in an algebraic number field and set $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \overline{\mathcal{O}})\}$.*
Then we have

1. *If $\# \mathrm{Pic}(\mathcal{O}) = 1$, then $\mathcal{O}$ is half-factorial.*
2. *If $\# \mathrm{Pic}(\mathcal{O}) \geq 3$, then $(\mathsf{D}(\mathrm{Pic}(\mathcal{O})))^2 \geq \mathsf{c}(\mathcal{O}) \geq 3$, $\min \triangle(\mathcal{O}) = 1$, and $\rho(\mathcal{O}) > 1$.*
3. *If $\# \mathrm{Pic}(\mathcal{O}) = 2$, then $\rho(\mathcal{O}) \leq 2$, $2 \leq \mathsf{c}(\mathcal{O}) \leq 4$, and $\min \triangle(\mathcal{O}) \leq 1$.*
   If, additionally, all localizations of $\mathcal{O}$ are finitely primary monoids of exponent 1, then, setting $k = \#\{\mathfrak{p} \in \mathcal{P}^* \mid [\overline{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\mathrm{Pic}(\mathcal{O})} = \mathrm{Pic}(\mathcal{O})\}$, it follows that
   - *$\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = \mathsf{c}(\mathcal{O}) = 2 + \min\{2, k\} \in \{2, 3, 4\}$;*
   - *$\rho(\mathcal{O}) = \frac{1}{2}\mathsf{c}(\mathcal{O}) \in \{1, \frac{3}{2}, 2\}$;*
   - *$\triangle(\mathcal{O}) = [1, \mathsf{c}(\mathcal{O}) - 2] \subset [1, 2]$;*

   and the following are equivalent:
   - *$\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = 2$.*
   - *$\mathsf{c}(\mathcal{O}) = 2$.*
   - *$\mathcal{O}$ is half-factorial.*

   If, additionally, $[\mathfrak{p}] = \mathbf{0}_{\mathrm{Pic}(\mathcal{O})}$ for all $\mathfrak{p} \in \mathcal{P}^*$, then the following is also equivalent:
   - *$\mathsf{t}(\mathcal{O}) = 2$.*

In particular, $\min \triangle(\mathcal{O}) \leq 1$ always holds.

Proof. By Lemma 3.1.17, there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^\times = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1, 2\}$ for all $i \in [1, r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$, $\mathcal{I}^*(\mathcal{O}) \cong D$, $\mathcal{O}_{\mathrm{red}}^\bullet \subset D$ is a saturated submonoid, $\mathrm{Pic}(\mathcal{O}) = \mathsf{q}(D/\mathcal{O}_{\mathrm{red}}^\bullet)$ is its class group, and each class contains some $p \in P$.

1. If $\# \operatorname{Pic}(\mathcal{O}) = 1$, then the assertion follows by Lemma 3.1.9.2.
2. If $\# \operatorname{Pic}(\mathcal{O}) \geq 3$, then the assertion follows by Lemma 3.1.9.3.
3. If $\# \operatorname{Pic}(\mathcal{O}) = 2$, then $\rho(\mathcal{O}) \leq 2$ and $2 \leq \mathsf{c}(\mathcal{O}) \leq 4$ by Lemma 3.1.9.4. If, additionally, all localizations of $\mathcal{O}$ are finitely primary monoids of exponent 1, then, by Lemma 3.1.17, we have $k_i = 1$ for all $i \in [1, r]$. If $k = 0$, then we are in the situation of Proposition 3.1.13.1 and thus $\mathcal{O}$ is half-factorial, $\mathsf{c}(\mathcal{O}) = 2$, and $\triangle(\mathcal{O}) = \emptyset$. If $k \geq 2$, then we are in the situation of Proposition 3.1.13.4, and thus $\rho(\mathcal{O}) = 2$, $\mathsf{c}(\mathcal{O}) = 4$, and $\triangle(\mathcal{O}) = \{1, 2\}$. If $k = 1$, then we are in the situation of Proposition 3.1.13.3, and thus $\rho(\mathcal{O}) \geq \frac{3}{2}$, $\mathsf{c}(\mathcal{O}) = 3$, and $\triangle(\mathcal{O}) = \{1\}$. Since $k_i = 1$ for all $i \in [1, r]$, we may use Proposition 3.1.13.5, and thus we find $\rho(\mathcal{O}) = \frac{3}{2}$ if $k = 1$ and $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = \mathsf{c}(\mathcal{O})$ in all cases. Putting all this together, we obtain the formulas in the assertion. The equivalence of the four statements follows by Corollary 3.1.16.

In particular, in all situations, we find $\min \triangle(\mathcal{O}) \leq 1$. □

In the case of quadratic and cubic number fields, we can do even better. First, we recall and reformulate a definition and the key result from [**18**].
Let $\mathcal{O}$ be an order in an algebraic number field and $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$. Then we call $\mathcal{O}_\mathfrak{p}$ a local order. Now let $\mathcal{O}_\mathfrak{p}$ be a local order such that its integral closure $\overline{\mathcal{O}_\mathfrak{p}}$ is local too. Now we fix the following notions. We denote by $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ the maximal ideal of $\mathcal{O}_\mathfrak{p}$ respectively $\overline{\mathcal{O}_\mathfrak{p}}$, by $k = \mathcal{O}_\mathfrak{p}/\mathfrak{m}$ and $\overline{k} = \overline{\mathcal{O}_\mathfrak{p}}/\overline{\mathfrak{m}}$ the residue class fields, and by $\pi : \overline{\mathcal{O}_\mathfrak{p}} \to \overline{k}$ the canonical homomorphism. For a prime $p \in \overline{\mathcal{O}_\mathfrak{p}}$ and $i \in \mathbb{N}$, we set

$$U_{i,p}(\mathcal{O}_\mathfrak{p}) = \{\varepsilon \in \overline{\mathcal{O}_\mathfrak{p}}^\times \mid \varepsilon p^i \in \mathcal{O}_\mathfrak{p}\} \quad \text{and} \quad V_{i,p}(\mathcal{O}_\mathfrak{p}) = \pi(U_{i,p}(\mathcal{O}_\mathfrak{p})) \cup \{0\},$$

as in [**18**]. Then $V_{i,p}(\mathcal{O}_\mathfrak{p})$ is a k-subspace of $\overline{k}$ by [**18**].

Lemma 3.1.19 ([**18**, Theorem 3.3]). *Using the above notions, the following are equivalent:*

1. *$\mathcal{O}_\mathfrak{p}$ is half-factorial.*
2. *$(U_{1,p}(\mathcal{O}_\mathfrak{p}))^2 = \overline{\mathcal{O}_\mathfrak{p}}^\times$.*
3. *$\{xy \mid (x, y) \in V_{1,p}(\mathcal{O}_\mathfrak{p}) \times V_{1,p}(\mathcal{O}_\mathfrak{p})\} = \overline{k}$.*

Lemma 3.1.20. *Let $\mathcal{O}$ be an order in an algebraic number field and $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ such that $\mathcal{O}_\mathfrak{p}$ is half-factorial.*

1. *$\overline{\mathcal{O}_\mathfrak{p}}$ is local and every atom of $\mathcal{O}_\mathfrak{p}$ is a prime of $\overline{\mathcal{O}_\mathfrak{p}}$.*
2. *Let $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ be the maximal ideals of $\mathcal{O}_\mathfrak{p}$ respectively $\overline{\mathcal{O}_\mathfrak{p}}$ and let $k = \mathcal{O}_\mathfrak{p}/\mathfrak{m}$ and $\overline{k} = \overline{\mathcal{O}_\mathfrak{p}}/\overline{\mathfrak{m}}$ be the residue class fields.*
   If $\dim_k \overline{k} \leq 3$, then $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}}^\bullet$ is a finitely primary monoid of exponent 1.
   In particular, if $\mathcal{O}$ is an order in a quadratic or cubic number field, then $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}}^\bullet$ is a finitely primary monoid of exponent 1.

Whenever $\mathcal{O}_\mathfrak{p}$ is a Cohen-Kaplansky domain, i.e., whenever it has up to units only finitely many atoms, the result from Lemma 3.1.20.1 can be found in [**1**, Theorem 6.3].

Proof of Lemma 3.1.20.

1. The assertion follows by Lemma 3.1.3.1.

2. By part 1, $\overline{\mathcal{O}_\mathfrak{p}}$ is local too. Thus $\mathfrak{m}$ respectively $\overline{\mathfrak{m}}$ is well-defined and k respectively $\overline{k}$ is a field. Since $\overline{\mathcal{O}_\mathfrak{p}}$ has up to units only one prime element by part 1 we write $V_1(\mathcal{O}_\mathfrak{p})$ instead of $V_{1,p}(\mathcal{O}_\mathfrak{p})$ and $U_1(\mathcal{O}_\mathfrak{p})$ instead of $U_{1,p}(\mathcal{O}_\mathfrak{p})$. Furthermore, we find $\mathcal{U}_1(\mathcal{O}_\mathfrak{p}^\bullet) = U_1(\mathcal{O}_\mathfrak{p})$. For short, we write $m = \dim_k \overline{k}$, $n = \dim_k V_1(\mathcal{O}_\mathfrak{p})$, and $q = \#k$. Now we distinguish three cases by m.
   Case 1 $m = 1$. Here $k = \overline{k}$ and therefore $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$. Thus $U_1(\mathcal{O}_\mathfrak{p}) = \overline{\mathcal{O}_\mathfrak{p}}^\times$ by [**18**, Lemma 3.2], and therefore $\mathcal{O}_\mathfrak{p}^\bullet \subset \overline{\mathcal{O}_\mathfrak{p}^\bullet}$ is of exponent 1 by the additional statement of Lemma 3.1.3.1.
   Case 2 $m = 2$. If $n = 1$, then $V_1(\mathcal{O}_\mathfrak{p}) = k$, and therefore $V_1(\mathcal{O}_\mathfrak{p}) * V_1(\mathcal{O}_\mathfrak{p}) = k \neq \overline{k}$, a contradiction to Lemma 3.1.19.3. If $n = 2$, then $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$, and the assertion follows as in Case 1.
   Case 3 $m = 3$. If $n = 1$, then we find the same contradiction as in Case 2 when $n = 1$ there. If $n = 2$, then $\#(V_1(\mathcal{O}_\mathfrak{p}) * V_1(\mathcal{O}_\mathfrak{p})) < q^3 = \#\overline{k}$ by [**18**, Lemma 2.5]. This is again a contradiction to Lemma 3.1.19.3. If $n = 3$, then $V_1(\mathcal{O}_\mathfrak{p}) = \overline{k}$, and the assertion follows as in Case 1.
   Let K be the algebraic number field containing $\mathcal{O}$. Then we find $m \leq [K : \mathbb{Q}]$ and the assertion follows. □

Now we can prove a slightly refined version of Theorem 3.1.18 for orders in quadratic and cubic number fields.

COROLLARY 3.1.21. *Let $\mathcal{O}$ be a non-principal locally half-factorial order in a quadratic or cubic number field and set $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \overline{\mathcal{O}})\}$.*
Then we have

1. *If $\#\operatorname{Pic}(\mathcal{O}) = 1$, then $\mathcal{O}$ is half-factorial.*
2. *If $\#\operatorname{Pic}(\mathcal{O}) \geq 3$, then $(\mathsf{D}(\operatorname{Pic}(\mathcal{O})))^2 \geq \mathsf{c}(\mathcal{O}) \geq 3$, and $\min \triangle(\mathcal{O}) = 1$.*
3. *If $\#\operatorname{Pic}(\mathcal{O}) = 2$, then, setting $k = \#\{\mathfrak{p} \in \mathcal{P}^* \mid [\overline{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = \operatorname{Pic}(\mathcal{O})\}$, it follows that*
   - $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(\mathcal{O}) = 2 + \min\{2, k\} \in \{2, 3, 4\}$*;*
   - $\rho(\mathcal{O}) = \frac{1}{2}\mathsf{c}(\mathcal{O}) \in \{1, \frac{3}{2}, 2\}$*;*
   - $\triangle(\mathcal{O}) = [1, \mathsf{c}(\mathcal{O}) - 2] \subset [1, 2]$.

In particular, $\min \triangle(\mathcal{O}) \leq 1$ always holds, and the following are equivalent:

- $\mathsf{c}_{\mathrm{mon}}(\mathcal{O}) = 2$.
- $\mathsf{c}(\mathcal{O}) = 2$.
- *$\mathcal{O}$ is half-factorial.*

If, additionally, $[\mathfrak{p}] = \mathbf{0}_{\operatorname{Pic}(\mathcal{O})}$ for all $\mathfrak{p} \in \mathcal{P}^*$—this is always true if $\#\operatorname{Pic}(\mathcal{O}) = 1$ or if $\mathcal{O}$ is an order in a quadratic number field—then the following is also equivalent:

- $\mathsf{t}(\mathcal{O}) = 2$.

PROOF. Part 1 respectively part 2 follows immediately from Theorem 3.1.18.1 respectively Theorem 3.1.18.2. By Lemma 3.1.20.2, all localizations $\mathcal{O}_\mathfrak{p}$ for $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ are finitely primary monoids of exponent (at most) 1. Thus part 3 follows by the additional statement of Theorem 3.1.18.3.
Now we prove the additional statement. First note $\min \triangle(\mathcal{O}) \leq 1$ follows by the additional statement of Theorem 3.1.18. If $\#\operatorname{Pic}(\mathcal{O}) \geq 3$, then none of the equivalent conditions

holds by part 2. If $\#\operatorname{Pic}(\mathcal{O}) = 2$, then the four equivalent conditions are shown in the additional statement of Theorem 3.1.18.3. If $\#\operatorname{Pic}(\mathcal{O}) = 1$, then $\mathcal{O} \cong \mathcal{I}^*(\mathcal{O})$, and therefore $\mathcal{O}$ is half-factorial. By Lemma 3.1.17 and Lemma 3.1.20.2, there is a monoid D, a set of prime elements $P \subset D$, $r \in \mathbb{N}$, and reduced half-factorial but not factorial monoids $D_i \subset [p_i] \times \widehat{D_i}^\times = \widehat{D_i}$ of type $(1, k_i)$ with $k_i \in \{1,2\}$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$ and $\mathcal{I}^*(\mathcal{O}) \cong D$. Now the other equivalent conditions follow by Lemma 3.1.3.2. □

If we compare the equivalent conditions in Corollary 3.1.21 for non-principal locally half-factorial orders in quadratic or cubic number fields with the ones given in [**14**, Theorem 1.7.3.6]—see below—for principal orders in algebraic number fields, we see that at least these special non-principal orders behave nearly the same as the principal ones.

THEOREM 3.1.22 (cf. [**14**, Theorem 1.7.3.6]). *Let $\mathcal{O}$ be a principal order in a quadratic or cubic number field.*
Then the following are equivalent.

1. *$\mathcal{O}$ is half-factorial.*
2. *$\#\operatorname{Pic}(\mathcal{O}) \le 2$.*
3. *$\mathsf{t}(\mathcal{O}) \le 2$.*
4. *$\mathsf{c}(\mathcal{O}) \le 2$.*

By Corollary 3.1.21.3, we get an additional bound on the elasticity of a non-principal order $\mathcal{O}$ in a quadratic or cubic number field such that its conductor is an inert prime ideal, say $(\mathcal{O} : \overline{\mathcal{O}}) = \mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ and $\mathfrak{p}\overline{\mathcal{O}} \in \mathfrak{X}(\overline{\mathcal{O}})$, in fact $\rho(\mathcal{O}) \le \frac{3}{2}$. Now we revisit the example from [**16**, example at the end of the publication]: Let $\mathcal{O} = \mathbb{Z}[3i]$. Then $\overline{\mathcal{O}} = \mathbb{Z}[i]$, $\#\operatorname{Pic}(\mathcal{O}) = 2$, $\mathcal{O}$ is locally half-factorial, and $(\mathcal{O} : \overline{\mathcal{O}}) = 3\mathcal{O} \in \mathfrak{X}(\mathcal{O})$ is an inert prime ideal in $\overline{\mathcal{O}}$. We set $\beta = 1 + 2i$ and $\beta' = 1 - 2i$. Then 3β, $3\beta'$, 3, and 5 are irreducible elements of $\mathcal{O}$ satisfying $(3\beta)(3\beta') = 3^2 \cdot 5$; thus $\rho(\mathcal{O}) \ge \frac{3}{2}$. Now we have equality by Corollary 3.1.21.3.

3.1.3. Localizations of half-factorial orders.

PROPOSITION 3.1.23. *Let D be a monoid, $P \subset D$ be a set of prime elements, and let $T \subset D$ be a reduced atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $D_1 \subset T$ be a divisor-closed submonoid and let $D_1 \subset [p] \times \widehat{D_1}^\times = \widehat{D_1}$ be a finitely primary monoid of rank 1 and exponent k. Let $H \subset D$ be a saturated half-factorial submonoid, $G = \mathsf{q}(D/H)$ its class group, and let each class in G contain some $p' \in P$.*
Then $\#G \le 2$ and D_1 is either

- *half-factorial or*
- *$\#G = 2$, say $G = \{\mathbf{0}, g\}$, $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1, 2\}$, $[p]_{D/H} = g$, and $[\varepsilon]_{D/H} = \mathbf{0}$ for all $\varepsilon \in \widehat{D_1}^\times$.*

PROOF. Define a homomorphism $\iota : T \to G$ by $\iota(t) = [t]_{D/H}$. Throughout the proof we write $\mathcal{B} = \{S \in \mathcal{B}(G, T, \iota) \mid \mathbf{0} \nmid S\}$ as in Lemma 1.2.16.4. If $\#G \ge 3$, then it follows by Lemma 1.2.17.1 that H is not half-factorial. If $\#G = 1$, then $H = D$ and, obviously, the first case of the assertion holds. Now, let $\#G = 2$, say $G = \{\mathbf{0}, g\}$. Since H is half-factorial, $\mathcal{B}$ is also half-factorial by Lemma 1.2.16. By Lemma 3.1.6, $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$ is equivalent to D_1 half-factorial. We show that either

- $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$ or
- $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1, 2\}$, $\iota(p) = g$, and $\iota(\widehat{D_1}^\times) = \{\mathbf{0}\}$.

If $\#\mathsf{v}_p(\mathcal{A}(D_1)) = 1$, i.e., $\mathsf{v}_p(\mathcal{A}(D_1)) = \{n\}$, then we find $n = 1$ since $N_{\geq k} \subset n\mathbb{N}_0$. Suppose we have $\#\mathsf{v}_p(\mathcal{A}(D_1)) > 1$. Then there are $n = \min \mathsf{v}_p(\mathcal{A}(D_1))$ and $m = \max \mathsf{v}_p(\mathcal{A}(D_1)) > n$. Let $\varepsilon, \eta \in \widehat{D_1}^\times$ be such that $p^n\varepsilon$, $p^m\eta \in \mathcal{A}(D_1)$. Now we distinguish four cases by $\iota(p^n\varepsilon)$ and $\iota(p^m\eta)$.

Case 1 $\iota(p^n\varepsilon) = \iota(p^m\eta) = \mathbf{0}$. Then $p^n\varepsilon$, $p^m\eta \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta)^k = (p^n\varepsilon)^k(p^{(m-n)k}\eta^k\varepsilon^{-k}).$$

There are k atoms on the left side and at least $k+1$ on the right side; clearly a contradiction to $\mathcal{B}$ half-factorial.

Case 2 $\iota(p^n\varepsilon) = \iota(p^m\eta) = g$. Then $p^n\varepsilon g$, $p^m\eta g \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta g)^k = (p^n\varepsilon g)^k(p^{(m-n)k}\eta^k\varepsilon^{-k})$$

There are k atoms on the left side and at least $k+1$ on the right side; clearly a contradiction to $\mathcal{B}$ half-factorial.

Case 3 $\iota(p^n\varepsilon) = \mathbf{0}$ and $\iota(p^m\eta) = g$. Then $p^n\varepsilon$, $p^m\eta g \in \mathcal{A}(\mathcal{B})$, and we find

$$(p^m\eta g)^k = \begin{cases} (p^n\varepsilon)^k(g^2)^{\frac{k}{2}}(p^{(m-n)k}\eta^k\varepsilon^{-k}) & k \text{ even} \\ (p^n\varepsilon)^k(g^2)^{\frac{k-1}{2}}(p^{(m-n)k}\eta^k\varepsilon^{-k}g) & k \text{ odd.} \end{cases}$$

There are k atoms on the left side and, in both cases, at least $k + 1 + \frac{k-1}{2}$ on the right side; clearly a contradiction to $\mathcal{B}$ half-factorial.

Case 4 $\iota(p^n\varepsilon) = g$ and $\iota(p^m\eta) = \mathbf{0}$. Then $p^n\varepsilon g$, $p^m\eta \in \mathcal{A}(\mathcal{B})$. Now we must again distinguish four cases.

Case 4.1 $2n < m$ and k even. Here we find

$$(g^2)^{\frac{k}{2}}(p^m\eta)^k = (p^n\varepsilon g)^k(p^{(m-n)k}\varepsilon^{-k}\eta^k).$$

There are $\frac{5}{2}k$ atoms on the left side and at least $k + \left\lceil \frac{(m-n)k}{m} \right\rceil$ atoms on the right side. This is a contradiction to $\mathcal{B}$ half-factorial, since $m > 2n$ by assumption.

Case 4.2 $2n < m$ and k odd. Here we find

$$(g^2)^{\frac{k+1}{2}}(p^m\eta)^{k+1} = (p^n\varepsilon g)^{k+1}(p^{(m-n)(k+1)}\varepsilon^{-k-1}\eta^{k+1}).$$

This leads to a contradiction as in the case where k was even.

Case 4.3 $m < 2n$ and k even. We choose $l \in \mathbb{N}$ maximal with $lm \leq (n-1)k$, and we find

$$(p^n\varepsilon g)^k = (g^2)^{\frac{k}{2}}(p^{nk-lm}\varepsilon^k\eta^{-l})(p^m\eta)^l.$$

There are k atoms on the left side and at least $\frac{k}{2} + \left\lceil \frac{nk-lm}{m} \right\rceil + l$ on the right. This is a contradiction to $\mathcal{B}$ half-factorial, since $m < 2n$ by assumption.

Case 4.4 $m < 2n$ and k odd. We choose $l \in \mathbb{N}$ maximal with $lm \leq (n-1)(k+1)$, and we find a contradiction to $\mathcal{B}$ half-factorial by looking at

$$(p^n\varepsilon g)^{k+1} = (g^2)^{\frac{k+1}{2}}(p^{n(k+1)-lm}\varepsilon^{k+1}\eta^{-l})(p^m\eta)^l.$$

Case 4.5 $m = 2n$. In this particular case, we must again handle two additional cases.

Case 4.5.1 $n > 1$. Then there is $n' \in (n, 2n)$ and $\gamma \in \widehat{D}_1^\times$ such that $p^{n'}\gamma \in \mathcal{A}(D_1)$. If $\iota(p^{n'}\gamma) = \mathbf{0}$, then the assertion follows with $p^{n'}\gamma$ and $p^m\eta$ as in Case 1. If $\iota(p^{n'}\gamma) = g$, then the assertion follows with $p^n\varepsilon$ and $p^{n'}\gamma$ as in Case 2.

Case 4.5.2 $n = 1$. Then $m = 2n = 2$. Without loss of generality we may assume that $p \in D_1$. Furthermore, $\iota(p^2\eta) = \mathbf{0}$ implies $\iota(\eta) = \mathbf{0}$. For the moment, we assume that $\iota(p) = \mathbf{0}$ and $\iota(\varepsilon) = g$. Then we are done by Case 1 with p and $p^2\eta$. If now $\iota(p) = g$ and $\iota(\varepsilon) = \mathbf{0}$, we show $\iota(\widehat{D_1}^\times) = \{\mathbf{0}\}$ or $\mathsf{v}_p(\mathcal{A}(D_1)) = \{1\}$. If $\iota(\widehat{D_1}^\times) = \{\mathbf{0}\}$, then the second case in the assertion is fulfilled. Now suppose $\iota(\widehat{D_1}^\times) = G$, say there is some $\gamma \in \widehat{D_1}^\times$ with $\iota(\gamma) = g$. Then there is some $k' \in [1, k]$ such that $p^{k'}\gamma \in D_1$. Thus there are $\varepsilon_1, \ldots, \varepsilon_l, \eta_1, \ldots, \eta_{l'} \in \widehat{D_1}^\times$ such that $(p\varepsilon_1) \cdot \ldots \cdot (p\varepsilon_l)(p^2\eta_1) \cdot \ldots \cdot (p^2\eta_{l'}) = p^{k'}\gamma$ is a factorization of $p^{k'}\gamma$ in D_1. Thus $\varepsilon_1 \cdot \ldots \cdot \varepsilon_l\eta_1 \cdot \ldots \cdot \eta_{l'} = \gamma$, and therefore either $\iota(\varepsilon_i) = g$ for some $i \in [1, l]$ or $\iota(\eta_j) = g$ for some $j \in [1, l']$. In the first case, we are in the situation of Case 1 with $p\varepsilon_i$ and $p^2\eta$, and in the second case, we are in the situation of Case 2 with p and $p^2\eta_j$. $\square$

Corollary 3.1.24. *Let $\mathcal{O}$ be a half-factorial order in an algebraic number field K, $\mathcal{O}_K$ is integral closure, and let $\mathfrak{p} \in \mathfrak{X}(\mathcal{O})$ be a prime ideal of $\mathcal{O}$ such that $\mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)$. Then $\#\operatorname{Pic}(\mathcal{O}) \le 2$ and $\mathcal{O}_\mathfrak{p}$ is either*

- *half-factorial, and $\mathcal{O}_\mathfrak{p}^\bullet \subset (\mathcal{O}_K)_\mathfrak{p}^\bullet$ is a half-factorial monoid of type $(1, k)$ with $k \in \{1, 2\}$, or*
- *$\mathfrak{p}$ ramifies in $\mathcal{O}_K$ with ramification degree 2, i.e. there is some $\overline{\mathfrak{p}} \in (\mathcal{O}_K)_\mathfrak{p}$ prime such that $\overline{\mathfrak{p}}^2 \sim \mathfrak{p}$.*

In particular, if K is a quadratic number field, then $\mathcal{O}_\mathfrak{p}$ is half-factorial.

Proof. Let $\mathcal{O}$ be a half-factorial order in an algebraic number field K, let $\mathcal{O}_K$ be its integral closure, let $\mathcal{P} = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \not\supset (\mathcal{O} : \mathcal{O}_K)\}$, and let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)\}$. By [**14**, Theorem 3.7.1], we find that

$$\mathcal{O}_{\mathrm{red}}^\bullet \subset \mathcal{F}(\mathcal{P}) \times T \quad \text{with } T = \prod_{\mathfrak{p} \in \mathcal{P}^*} (\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}}$$

is a saturated cofinal submonoid with class group $\operatorname{Pic}(\mathcal{O})$. Now, we obtain $\#\operatorname{Pic}(\mathcal{O}) \le 2$ by Lemma 1.2.17.1. Since $\mathcal{O}$ is half-factorial, i.e., $\rho(\mathcal{O}) = 1 < \infty$, we find, by [**16**, Corollary 4.i], that $\mathfrak{p}$ does not split in $\mathcal{O}_K$. Thus $(\mathcal{O}_K)_\mathfrak{p}$ is a discrete valuation domain, in particular, it is local, and thus $\mathcal{O}_\mathfrak{p}^\bullet \subset (\mathcal{O}_K)_\mathfrak{p}^\bullet$ is a finitely primary monoid of rank 1. Since $(\mathcal{O}_\mathfrak{p}^\bullet)_{\mathrm{red}} \subset T$ is a divisor-closed submonoid, the assertion follows immediately by Proposition 3.1.23.
If K is a quadratic number field, then $\mathcal{O}$ half-factorial implies that $\mathfrak{p}$ is inert by [**14**, First paragraph in the Proof of A2 in the Proof of Theorem 3.7.15], and therefore $\mathcal{O}_\mathfrak{p}$ is half-factorial. $\square$

3.1.4. Characterization of half-factorial orders in quadratic number fields.

Corollary 3.1.25. *Let $\mathcal{O}$ be a non-principal order in a quadratic number field K, let $\mathcal{O}_K$ be its integral closure, and let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\mathcal{O} : \mathcal{O}_K)\}$.*
Then the following are equivalent:

1. *$\mathcal{O}$ is half-factorial.*
2. *$\mathsf{c}(\mathcal{O}) = 2$.*
3. *$\#\operatorname{Pic}(\mathcal{O}) \le 2$, $\mathcal{O}$ is locally half-factorial and, for all $\mathfrak{p} \in \mathcal{P}^*$, $[(\mathcal{O}_K)_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times]_{\operatorname{Pic}(\mathcal{O})} = [\mathbf{0}]_{\operatorname{Pic}(\mathcal{O})}$.*
4. *$\#\operatorname{Pic}(\mathcal{O}) \le 2$ and, for all $\mathfrak{p} \in \mathcal{P}^*$,*

- $[(\mathcal{O}_K)_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times]_{\mathrm{Pic}(\mathcal{O})} = [\mathbf{0}]_{\mathrm{Pic}(\mathcal{O})}$,
- $\mathfrak{p}$ *is inert in* $\mathcal{O}_K$*, and*
- $\mathfrak{p}^2 \not\supset (\mathcal{O} : \mathcal{O}_K)$.

PROOF. **1** $\Leftrightarrow$ **2** By Corollary 3.1.24, we have $\mathcal{I}^*(\mathcal{O})$ half-factorial. Thus the assertion is already shown in the additional statement of Corollary 3.1.21.
1 $\Rightarrow$ **3** By Corollary 3.1.24, $\#\,\mathrm{Pic}(\mathcal{O}) \le 2$ and $\mathcal{O}_\mathfrak{p}$ is half-factorial for all $\mathfrak{p} \in \mathcal{P}^*$. Now we get $[(\mathcal{O}_K)_\mathfrak{p}^\times/\mathcal{O}_\mathfrak{p}^\times]_{\mathrm{Pic}(\mathcal{O})} = [\mathbf{0}]_{\mathrm{Pic}(\mathcal{O})}$ by the same construction as in the proof of Corollary 3.1.21 and Theorem 3.1.18 using Proposition 3.1.13.
3 $\Rightarrow$ **1** Since, by assumption, $\mathcal{O}$ is locally half-factorial, this implication follows, directly, by the same construction as in the proof of Corollary 3.1.21 and Theorem 3.1.18 using Proposition 3.1.13.
3 $\Leftrightarrow$ **4** Since, for all $\mathfrak{p} \in \mathcal{P}^*$, $\mathcal{O}_\mathfrak{p}$ is half-factorial if and only if $\mathfrak{p}$ is inert in $\mathcal{O}_K$ and $\mathfrak{p}^2 \not\supset (\mathcal{O} : \mathcal{O}_K)$, the assertion follows. □

3.2. Non-principal locally half-factorial orders in algebraic number fields with cyclic class groups

3.2.1. Monoid-theoretic situation.

LEMMA 3.2.1. *Let D be a monoid, $P \subset D$ be a set of prime elements, $D_1 \subset D$ be an atomic submonoid, and $D_2 \subset [p] \times \widehat{D_2}^\times = \widehat{D_2}$ be a reduced monoid of type $(1,1)$ such that $D = \mathcal{F}(P) \times D_1 \times D_2$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ be its class group, suppose each class in G contains some $p \in P$, define a homomorphism $\iota : D_1 \times D_2 \to G$ by $\iota(t) = [t]_{D/H}$, and suppose $\#G \ge 2$ and $\iota(D_2) = [1]_{D/H}$.*
Then there is an atomic submonoid $H' \subset H$ such that $H' \subset D' = \mathcal{F}(P) \times D_1$ is a saturated submonoid with class group $\mathsf{q}(D'/H') = G$. Furthermore, $\mathsf{c}(H) = \mathsf{c}(H')$, $\rho(H) = \rho(H')$, and $\triangle(H) = \triangle(H')$.

PROOF. Since $\iota(D_2) = [1]_{D/H}$ we clearly have $D_2 \subset H$ and, since $H \subset D$ is a saturated submonoid and since there is a submonoid $D' = \mathcal{F}(P) \times D_1 \subset D$ such that $D = D' \times D_2$, there is some $H' \subset H$ such that $H = H' \times D_2$. Obviously, $H' \subset D'$ is also a saturated submonoid. By Lemma 1.2.18.3-5 and by Lemma 3.1.3, we have

$$\begin{aligned}
&\mathsf{c}(H) = \sup\{\mathsf{c}(H'), \mathsf{c}(D_2)\} = \sup\{\mathsf{c}(H'), 2\} = \mathsf{c}(H');\\
&\rho(H) = \sup\{\rho(H'), \rho(D_2)\} = \sup\{\rho(H'), 1\} = \rho(H');\, and\\
&\triangle(H') \cup \triangle(D_2) \subset \triangle(H),
\end{aligned}$$

and, since $\triangle(D_2) = \emptyset$, equality holds. □

LEMMA 3.2.2. *Let $m \in \mathbb{N}_{\ge 2}$, $n \in \mathbb{N}$, $k_1, \ldots, k_m \in \mathbb{N}$, $k \in (0, n)$ be such that*

$$\sum_{i=1}^m k_i + k \equiv 0 \mod n \quad and \quad \sum_{i=1}^m k_i > n.$$

Then there are

$$\emptyset \ne I \subsetneq [1, m] \text{ and } k' \in [0, k] \text{ such that } \sum_{i \in I} k_i + k' \equiv 0 \mod n.$$

Proof. Since $\sum_{i=1}^{m} k_i + k \equiv 0 \mod n$ and $k \in (0,n)$, clearly, $\sum_{i=1}^{m} k_i \not\equiv 0 \mod n$. Now we proceed by induction on m. If $m = 2$, we have $k_1+k_2 > jn$ and $k_1+k_2+k = (j+1)n$ for some $j \in \mathbb{N}$. Now we choose $k_1', k_2' \in [0,k)$ and $j_1, j_2 \in \mathbb{N}$ with $j_1 + j_2 = j+1$ such that $k_i = j_i n - k_i'$ for $i = 1, 2$. Then we have $j_1 n - k_1' + j_2 n - k_2' + k = (j+1)n$, and therefore $k = k_1' + k_2'$. Hence, we get the assertion with $I = \{1\} \subsetneq \{1,2\}$ and $k' = k_1' \le k$.
Now let $m > 2$. To proceed from m to $m+1$, we can simply use the base case on $\sum_{i=1}^{m} k_i$ and k_{m+1}. □

Lemma 3.2.3. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ reduced but not factorial monoids of type $(1,1)$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ be its class group cyclic of order $n \ge 2$, suppose each class in G contains some $p \in P$, define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$, and let $s \in [1,r]$ be such that $\iota(p_i) = [1]_{D/H}$ for all $i \in [1,s]$ and $\iota(D_i) = [1]_{D/H}$ for all $i \in [s+1,r]$.
Then we have*

$$\begin{aligned}
&\mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)) = \\
&\quad \Bigg\{ S \prod_{i \in I} p_i \varepsilon_i \,\Bigg|\, S \in \mathcal{F}(G),\ I \subset [1,s],\ \varepsilon_i \in \widehat{D_i}^{\times},\ \sum_{i\in I} \iota(p_i\varepsilon_i) + \sigma(S) = \mathbf{0},\ \text{and} \\
&\quad \sum_{i \in J} \iota(p_i \varepsilon_i) + \sigma(S') \ne \mathbf{0} \text{ for all } J \subset I \text{ and} \\
&\quad S' \mid S \text{ without } J = I \text{ and } S' = S \text{ at the same time} \Bigg\} \\
&\quad \cup \left\{ p_i \varepsilon_i \,\middle|\, i \in [s+1,r],\ \varepsilon_i \in \widehat{D_i}^{\times} \right\}
\end{aligned}$$

In particular, if $a = g^k \prod_{i\in I} p_i\varepsilon_i \in \mathcal{A}(\mathcal{B}(G, D_1 \times \ldots \times D_r, \iota))$ with $k < n$, $I \subset [1,s]$, and $\varepsilon_i \in \widehat{D_i}^{\times}$. Then either

- $\sum_{i\in I} (G : \langle \iota(\varepsilon_i) \rangle) < n$ *or*
- $\sum_{i\in I} (G : \langle \iota(\varepsilon_i) \rangle) \equiv 0 \mod n$.

Proof. The main part of the lemma follows by Lemma 3.2.1 and the structure of the set atoms of a monoid of type $(1,1)$ given in Lemma 3.1.3. The additional statement follows by Lemma 3.2.2. □

Now we are able to give some non-trivial lower bounds on the elasticity.

Proposition 3.2.4. *Let D be a monoid, $P \subset D$ a set of prime elements, $r \in \mathbb{N}$, and $D_i \subset [p_i] \times \widehat{D_i}^{\times} = \widehat{D_i}$ reduced but not factorial monoids of type $(1,1)$ for all $i \in [1,r]$ such that $D = \mathcal{F}(P) \times D_1 \times \ldots \times D_r$. Let $H \subset D$ be a saturated submonoid, $G = \mathsf{q}(D/H)$ its class group cyclic of order $n \ge 2$, say $G = \langle g \rangle$, suppose each class in G contains some $p \in P$, define a homomorphism $\iota : D_1 \times \ldots \times D_r \to G$ by $\iota(t) = [t]_{D/H}$, and let $s \in [1,r]$ be such that $\iota(p_i) = [1]_{D/H}$ for $i \in [1,s]$ and $\iota(D_i) = [1]_{D/H}$ for $i \in [s+1,r]$.*

1. *$\rho(H) \ge \frac{\mathsf{D}(G)}{2}$.*

2. *Let* $\emptyset \neq I \subset [1,s]$ *be a non-empty subset such that* $\#I \le n$ *and* $\sum_{i\in I}(G : \iota(\widehat{D_i}^\times)) \le n$. *Then*
$$\rho(H) \ge \#I + 1 - \sum_{i\in I} \frac{(G : \iota(\widehat{D_i}^\times))}{\mathsf{D}(G)}.$$
In particular, the right hand side obtains its maximum whenever I *is maximal with respect to its cardinality and* $\sum_{i\in I}(G : \iota(\widehat{D_i}^\times))$ *is minimal.*
3. *Let* $\emptyset \neq J \subset [1,s]$ *be a non-empty subset such that* $\#J \le n$ *and, for* $j \in J$, *let* $\varepsilon_j \in \widehat{D_j}^\times \setminus \widehat{D_j}^\times_{\mathbf{0}}$ *be such that* $\sum_{j\in J}(G : \langle\iota(\varepsilon_j)\rangle) \equiv 0 \mod n$ *and there is no proper non-empty subset* $\emptyset \neq J' \subsetneq J$ *such that* $\sum_{j\in J'}(G : \langle\iota(\varepsilon_j)) \equiv 0 \mod n$.
Then
$$\rho(H) \ge \#J.$$
In particular, the right hand side obtains its maximum whenever J *is maximal with respect to its cardinality.*

If, in particular, $n \in \mathbb{P}$ *is prime and* $\emptyset \neq I \subset [1,s]$ *is a non-empty subset with* $\#I \le n$, *then we have*
$$n \ge \rho(H) \ge \begin{cases} \#I + 1 - \frac{\#I}{n} & \#I \ge \frac{n^2-2n}{2n-2}, \\ \frac{n}{2} & \frac{n^2-2n}{2n-2} \ge \#I. \end{cases}$$

Proof. By Lemma 3.2.1, we may without loss of generality assume that $s = r$, i.e. $\iota(p_i) = \mathbf{0}$ and $\iota(\widehat{D_i}^\times) \neq \{\mathbf{0}\}$ for all $i \in [1,r]$. For short, we write $\mathcal{B} = \mathcal{B}(G, D_1 \times \ldots \times D_r, \iota)$. By Lemma 1.2.16, we have $\rho(H) = \rho(\mathcal{B})$.

1. Since $\mathcal{B}(G) \subset \mathcal{B}$ is a divisor-closed submonoid, we have $\rho(\mathcal{B}) \ge \rho(\mathcal{B}(G)) = \frac{\mathsf{D}(G)}{2}$, by [**14**, Theorem 3.4.10.4].
2. For $i \in I$, let $\varepsilon_i \in \widehat{D_i}^\times$ be such that $\langle\iota(\varepsilon_i)\rangle = \iota(\widehat{D_i}^\times)$. Then there is some $k \in [0,n)$ such that
$$a = g^k \prod_{i\in I} p_i\varepsilon_i \in \mathcal{A}(\mathcal{B})$$
by Lemma 3.2.3. Now we calculate the elasticity of the n-th power of a. We find
$$a^n = (g^k \prod_{i\in I} p_i\varepsilon_i)^n = (g^n)^k \prod_{i\in I}(p_i\varepsilon_i^n)p_i^{n-1},$$
where g^n, $p_i\varepsilon_i$, $p_i \in \mathcal{A}(\mathcal{B})$ for all $i \in I$, and thus
$$\rho(a^n) \ge \frac{k + n\#I}{n} = \#I + \frac{k}{n} = \#I + \frac{n - \sum_{i\in I}(G : \iota(\widehat{D_i}^\times))}{n} = \#I + 1 - \sum_{i\in I}\frac{(G : \iota(\widehat{D_i}^\times))}{n}.$$
Since $\mathsf{D}(G) = n$, we are done.
3. For $j \in J$, let $\varepsilon_j \in \widehat{D_j}^\times$ be as in the assumption. Then we find
$$a = \prod_{j\in J} p_j\varepsilon_j \in \mathcal{A}(\mathcal{B}).$$
Now we calculate the elasticity of the n-th power of a. We find
$$a^n = \left(\prod_{j\in J} p_j\varepsilon_j\right)^n = \prod_{j\in J}(p_j\varepsilon_j^n)p_j^{n-1},$$

where g^n, $p_j\varepsilon_j^n$, $p_j \in \mathcal{A}(\mathcal{B})$ for all $j \in J$, and thus

$$\rho(a^n) = \frac{n\#J}{n} = \#J.$$

If now, in particular, $\#G = n \in \mathbb{P}$, then G has no non-trivial subgroups. Therefore $\iota(\widehat{D_i}^\times) = G$ for all $i \in [1,r]$, and thus we may assume that $\iota(p_i) = \mathbf{0}$ for all $i \in [1,r]$. Now we apply part 2 with $(G : \iota(\widehat{D_i}^\times)) = 1$ for all $i \in I$. Thus $\sum_{i\in I}(G : \iota(\widehat{D_i}^\times)) = \#I \le n$. For short, we set $\#I = k$. Now it remains to show that

$$\max\left\{k+1-\frac{k}{n}, \frac{n}{2}\right\} = \begin{cases} k+1-\frac{k}{n} & k \ge \frac{n^2-2n}{2n-2} \\ \frac{n}{2} & \frac{n^2-2n}{2n-2} \ge k. \end{cases}$$

This can be seen by the following easy calculation

$$\begin{aligned} k+1-\frac{k}{n} &\ge \frac{n}{2} \\ k(2n-2)+2n-n^2 &\ge 0 \\ k &\ge \frac{n^2-2n}{2n-2}. \end{aligned}$$

□

Corollary 3.2.5. *Let $\mathcal{O}$ be a non-principal locally half-factorial order in an algebraic number field such that all localizations of $\mathcal{O}$ are finitely primary monoids of exponent* 1 *and $\#\operatorname{Pic}(\mathcal{O}) = n \in \mathbb{P}$. Let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\mathcal{O}) \mid \mathfrak{p} \supset (\bar{\mathcal{O}} : \mathcal{O})\}$ and let $\iota : \mathsf{q}(\mathcal{O}) \to \operatorname{Pic}(\mathcal{O})$ be a homomorphism defined by $\iota(t) = [t]_{\operatorname{Pic}(\mathcal{O})}$ for all $t \in \mathsf{q}(\mathcal{O})$. Let $\emptyset \ne I \subset \mathcal{P}^*$ be a non-empty subset such that $\#I \le n$ and $\sum_{\mathfrak{p}\in I}(\operatorname{Pic}(\mathcal{O}) : \iota(\widehat{(}^\times\mathcal{O}_\mathfrak{p}))) \le n$.*
Then

$$n \ge \rho(\mathcal{O}) \ge \begin{cases} \#I+1-\frac{\#I}{n} & \#I \ge \frac{n^2-2n}{2n-2}, \\ \frac{n}{2} & \frac{n^2-2n}{2n-2} \ge \#I. \end{cases}$$

Proof. Clear by Proposition 3.2.4. □

With the last example in this section, we want to illustrate which (easy) subsets of the appearing T-block monoids are not half-factorial.

Example 3.2.6. Let D be an atomic monoid, $P \subset D$ be a set of prime elements, and $T \subset D$ be an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $D_1 \subset T$ be a divisor-closed submonoid and $D_1 \subset [p_1] \times \widehat{D_1}^\times = \widehat{D_1}$ a monoid of type $(1,1)$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ its class group cyclic of order $n \ge 2$, say $G = \langle g\rangle$, suppose each class in G contain some $p \in P$, define a homomorphism $\iota(T) \to G$ by $\iota(t) = [t]_{D/H}$, and let $\varepsilon \in \widehat{D_1}^\times$ be such that $\langle\iota(\varepsilon)\rangle = \iota(\widehat{D_1}^\times) \ne \{\mathbf{0}\}$.
Then the set

$$[\![\{g^k p_1\varepsilon \mid kg + \iota(p_1\varepsilon) = \mathbf{0}\}]\!]$$

is not half-factorial.
In particular, H is not half-factorial.

Proof. Let $n_1 \in [0,n)$ and $n_1' \in [1,n)$ be such that $\iota(p_1) = n_1 g$ and $\iota(\varepsilon) = n_1' g$. If $n_1' = 1$, we can choose p_1 in such a way that $n_1 = 0$. If $n_1 = 0$, then the assertion follows by the proof of Proposition 3.2.4. Since n_1, $n_1' \mid n$ we have n_1, $n_1' \le \frac{n}{2}$ and therefore $n_1 + n_1' \le n$. Now we distinguish 4 cases.

Case 1. $n_1 + n_1' = n$. Here we have $n_1 = n_1' = \frac{n}{2}$, and therefore $p_1^2 \in \mathcal{A}(\mathcal{B})$. Now we consider the factorizations

$$(p_1\varepsilon)^4 = (p_1\varepsilon^3)(p_1\varepsilon)(p_1^2)$$

and find 4 atoms on the left side and 3 on the right side.
Case 2. $n = 4$, $n_1 = 1$, and $n_1' = 2$. We find

$$(p_1\varepsilon g)^4 = (p_1\varepsilon^4 g^3)(p_1^3 g).$$

These are 4 atoms on the left side and 2 on the right side.
Case 3. $n = 6$ and $n_1 + n_1' = 5$. Here we find

$$(p_1\varepsilon g)^6 = \begin{cases} (p_1\varepsilon^6 g^3)(p_1 g^3)(p_1^2)^2 & n_1 = 3 \text{ and } n_1' = 2, \\ (p_1\varepsilon^6 g^4)(p_1^2 g^2)(p_1^3) & n_1 = 2 \text{ and } n_1' = 3. \end{cases}$$

These are 6 atoms on the left side and 4 respectively 3 on the right side.
Case 4. $n_1 + n_1' < n$ and neither case 2 nor case 3 happens. Thus we have $n - n_1 - n_1' - 1 \geq 1$ and we find

$$(p_1\varepsilon g^{n-n_1-n_1'})^n = (p_1\varepsilon^n g^{n-n_1})(p_1^{n-1} g^{n_1})(g^{n-n_1-n_1'-1})^n.$$

These are n atoms on the left side and at least $n+2$ atoms on the right side. $\square$

CHAPTER 4

$\min \triangle(R_f)$ and $\min \triangle(\mathcal{O}_{K,f})$

4.1. Extensions of discrete valuation domains

Extensions of discrete valuation domains have been studied in [**21**, Subsection 3.3] and some of the results presented here can be found there, too.

Definition 4.1.1. Let A be a discrete valuation domain with prime element p, K its quotient field, L/K a finite extension, and $\bar{A}$ the integral closure of A in L. (Then $\bar{A}$ is a semi-local principal ideal domain. Let $s \in \mathbb{N}$ an $\pi_1, \ldots, \pi_s$ be a system of pairwise non-associated prime elements of $\bar{A}$. Let $p = \pi_1^{e_1} \cdot \ldots \cdot \pi_s^{e_s} \varepsilon$ with $\varepsilon \in \bar{A}^\times$ and $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$ a prime decomposition of p in $\bar{A}$.) Furthermore we set $R = A + f\bar{A}$ with $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ and $f = \pi_1^{k_1} \cdot \ldots \cdot \pi_s^{k_s} \in \bar{A}$ and $\mathsf{v}_{\pi_i} : \bar{A}^\bullet \to \mathbb{N}_0$ the π_i-adic valuation of $\bar{A}^\bullet$ for $i = 1, \ldots, s$ and $\mathsf{v}_p : A^\bullet \to \mathbb{N}_0$ the p-adic valuation of $A^\bullet$. (Then we get $\mathsf{v}_{\pi_i}(p) = e_i$ and $\mathsf{v}_{\pi_i}(f) = k_i$ for all $i \in [1, s]$, and $(R : \bar{A}) = f\bar{A}$.) This situation ($A \subset R = A + f\bar{A} \subset \bar{A}$) is called an *extension of a discrete valuation domain.*

Proposition 4.1.2. *Let R be a one-dimensional local noetherian domain with maximal ideal $\mathfrak{m}$ such that its integral closure $\bar{R}$ is a finitely-generated R-module. (Then $\bar{R}$ is a semi-local Dedekind domain, thus a principal ideal domain.) Let $s \in \mathbb{N}$ and $\pi_1, \ldots, \pi_s \in \bar{R}$ be a system of pairwise non-associated prime elements of $\bar{R}$. Now we set $(R : \bar{R}) = \pi_1^{k_1} \cdot \ldots \cdot \pi_s^{k_s} \bar{R}$ with $k_1, \ldots, k_s \in \mathbb{N}$.*
Then $R^\bullet$ is a finitely primary monoid of exponent $k = \max\{k_1, \ldots, k_s\}$ and of rank s.
In particular, $\mathfrak{m}$ does not split in $\bar{R}$ if and only if $s = 1$.

Proof. Obviously, $\bar{R}^\bullet = [\pi_1, \ldots, \pi_s] \times \bar{R}^\times$. Since $\bar{R} \supset R$ is integral, we have $\bar{R}^\times \cap R = R^\times$ and $\pi_i \bar{R} \cap R = \mathfrak{m}$ for all $i \in [1, s]$. Let $a = \pi_1^{n_1} \cdot \ldots \cdot \pi_s^{n_s} \varepsilon \in \bar{R}^\bullet$ with $n_1, \ldots, n_s \in \mathbb{N}_0$ and $\varepsilon \in \bar{R}^\times$. If $n_i \geq k$ for all $i \in [1, s]$, then $a\bar{R} \subset (R : \bar{R}) \subset R$ and thus $a \in R$. Now it remains to prove: $R^\bullet \backslash R^\times \subset \pi_1 \cdot \ldots \cdot \pi_s \bar{R}^\bullet$. Since $\pi_1, \ldots, \pi_s$ are prime elements in $\bar{R}$, we have $\pi_1 \bar{R}^\bullet \cap \ldots \cap \pi_s \bar{R}^\bullet = \pi_1 \cdot \ldots \cdot \pi_s \bar{R}^\bullet$. Now the assertion follows from $R^\bullet \backslash R^\times = \mathfrak{m} \backslash \{0\} \subset \pi_i \bar{R}^\bullet$ for all $i \in [1, s]$. In particular, since $\pi_i \bar{R} \cap R = \mathfrak{m}$ for all $i \in [1, s]$, $\mathfrak{m}$ does not split in $\bar{R}$ if and only if $s = 1$. $\square$

Definition 4.1.3. Let $s \in \mathbb{N}$, $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$, $n \in \mathbb{N}$, and H be a monoid of type $(\mathbf{e}, n\mathbf{e})$.
We call H a *strict monoid of type* $(\mathbf{e}, n)$ if for all $a \in H \setminus H^\times$, $i \in [1, s]$, and $l \in [1, n-1]$

$$\mathsf{v}_{p_i}(a) = le_i \quad \text{implies} \quad \mathsf{v}_{p_j}(a) = le_j \text{ for all } j \in [1, s].$$

Now we can give some characterization of extensions of discrete valuation domains leading to strict monoids of type $(\mathbf{e}, n)$.

PROPOSITION 4.1.4. *Let $A \subset R = A + f\bar{A} \subset \bar{A}$ be an extension of a discrete valuation domain A with prime element p and $f \in A$.*
Then:

1. *R is a noetherian one-dimensional local domain with maximal ideal $pA + f\bar{A}$.*
2. *$R^\bullet$ is a monoid of type $(\mathbf{e}, \mathbf{k})$.*
3. *If $f \sim p^n$ for $n \in \mathbb{N}$, then $R^\bullet$ is a strict monoid of type $(\mathbf{e}, n)$.*

PROOF.

1. Let $\varphi : \bar{A} \to \bar{A}/f\bar{A}$ be the canonical epimorphism. Then we find $R = A + f\bar{A} = \varphi^{-1}(\varphi(A))$, and thus $R \subset \bar{A}$ is a subring. By the Theorem of Krull-Akizuki, R is a one-dimensional noetherian domain.
Let $\mathfrak{m}$ be a maximal ideal of R. Since $\bar{A} \supset R$ is integral, we find $\mathfrak{m} = \pi_i\bar{A} \cap R$ for all $i \in [1, s]$. Now, it is sufficient to show that $\pi_i\bar{A} \cap R = pA + f\bar{A}$ for all $i \in [1, s]$. We show $R \setminus \pi_i\bar{A} \subset R^\times$ for all $i \in [1, s]$. Let $x = a + fy \in R \setminus \pi_i\bar{A}$ with $a \in A$ and $y \in \bar{A}$, then $a \in A \setminus \pi_i\bar{A} = A \setminus fA = A^\times$. If we have $u \in A$ with $ua = 1$, then $ux = 1 + fuy \in 1 + f\bar{A}$, and therefore, there is no maximal ideal of $\bar{A}$ containing ux, and thus $ux \in \bar{A}^\times \cap R = R^\times$.
2. By part 1, we can apply Proposition 4.1.2 and we get that $R^\bullet \subset \bar{A}^\bullet$ is a finitely primary monoid of rank s and exponent $\max\{k_1, \ldots, k_s\}$. Let now $\mathsf{v}_p : A^\bullet \to N_0$ be the p-adic valuation of $A^\bullet$. Obviously, $\mathsf{v}_{\pi_i}(A^\bullet) = e_i\mathsf{v}_p(A^\bullet) = e_i\mathbb{N}_0$ and $\mathsf{v}_{\pi_i}(f\bar{A}^\bullet) = \mathbb{N}_{\geq k_i}$, for all $i \in [1, s]$. Thus we find $\mathsf{v}_{\pi_i}(R) \supset e_i\mathbb{N}_0 \cup \mathbb{N}_{\geq k_i}$, for all $i \in [1, s]$, and therefore it suffices to show thor all $i \in [1, s]$ that:
$$(a, \bar{A}) \in A^\bullet \times \bar{A}^\bullet \text{ and } \mathsf{v}_{\pi_i}(a + f\bar{a}) < k_i \quad \Rightarrow \quad \mathsf{v}_{\pi_i}(a + f\bar{a}) \in e_i\mathbb{N}_0\,.$$
Let $i \in [1, s]$ and $(a, \bar{a}) \in A^\bullet \times \bar{a}^\bullet$ such that $\mathsf{v}_{\pi_i}(a + f\bar{a}) < k_i$. Since $\mathsf{v}_{\pi_i}(a + f\bar{a}) \geq \min\{\mathsf{v}_{\pi_i}(a),\ \mathsf{v}_{\pi_i}(f\bar{a})\}$ and $\mathsf{v}_{\pi_i}(f\bar{a}) \geq k_i$, we have $\mathsf{v}_{\pi_i}(a) < \mathsf{v}_{\pi_i}(f\bar{a})$ and $\mathsf{v}_{\pi_i}(a + f\bar{a}) = \mathsf{v}_{\pi_i}(a) = e_i\mathsf{v}_p(a) \in e_i\mathbb{N}_0$. Thus $R^\bullet \subset \bar{A}^\bullet$ is a monoid of type $(\mathbf{e}, \mathbf{k})$.
3. Here we have $\mathbf{k} = n\mathbf{e}$. By part 2, we obtain that $R^\bullet$ is a monoid of type $(\mathbf{e}, n\mathbf{e})$. Now we must show the second property in Definition 4.1.3. Let $a = a' + fa'' \in R^\bullet$ with $a' \in A$, $a'' \in \bar{A}$, and $l \in [1, n)$ be such that $\mathsf{v}_{\pi_1}(a) = le_1$. Suppose $a' = 0$, then $\mathsf{v}_{\pi_1}(a) = \mathsf{v}_{\pi_1}(fa'') \geq k_1$, a contradiction. Suppose $a'' = 0$, then $a = a' \in A^\bullet = [p] \times A^\times$ and the assertion is obvious. Now we can suppose that $a',\ a'' \neq 0$. Since $k_1 = ne_1 > \mathsf{v}_{\pi_1}(a) \geq \min\{\mathsf{v}_{\pi_1}(a'),\ \mathsf{v}_{\pi_1}(fa'')\} = \min\{\mathsf{v}_{\pi_1}(a'),\ k_1 + \mathsf{v}_{\pi_1}(a'')\}$, we have $\mathsf{v}_{\pi_1}(a') < \mathsf{v}_{\pi_1}(fa'')$, and therefore $le_1 = \mathsf{v}_{\pi_1}(a) = \mathsf{v}_{\pi_1}(a')$, i.e. $a' = p^l\varepsilon$ with $\varepsilon \in A^\times$, and thus $\mathsf{v}_{\pi_j}(a') = le_j$ for all $j \in [1, s]$. Let now $j \in [1, s]$. Then $\mathsf{v}_{\pi_j}(a') = le_j < ne_j \leq \mathsf{v}_{\pi_j}(fa'')$, thus $\mathsf{v}_{\pi_j}(a) = \mathsf{v}_{\pi_j}(a') = le_j$. □

LEMMA 4.1.5. *Let $A \subset R = A + f\bar{A} \subset \bar{A}$ be an extension of a discrete valuation domain A with prime element p such that $\bar{A}$ is a discrete valuation domain with prime element π. Suppose that $p \sim \pi^e$ and $f \sim p^n$, where $e,\ n \in \mathbb{N}$.*
Then $en \in \mathsf{v}_\pi(\mathcal{A}(R))$.

PROOF. By Proposition 4.1.4.3, we have that $R^\bullet \subset \bar{A}^\bullet = [\pi] \times \bar{A}^\times$ is a strict monoid of type (e, en). By the minimality of en (in Definition 4.1.1 or Definition 3.1.1), it is obvious that there is $\alpha \in \bar{A}^\times$ such that $\pi^l\alpha \in R$ if and only if $l \geq en$. Now assume $\pi^{en}\alpha \notin \mathcal{A}(R)$. Then there are $n_1,\ n_2 \in [1, en)$ and $\varepsilon_1,\ \varepsilon_2 \in \bar{A}^\times$ such that $\pi^{n_1}\varepsilon_1,\ \pi^{n_1}\varepsilon_2 \in R$

and $(\pi^{n_1}\varepsilon_1)(\pi^{n_2}\varepsilon_2) = \pi^{en}\alpha$, i.e. $n_1 + n_2 = en$ and $\varepsilon_1\varepsilon_2 = \alpha$. Since $\pi^{n_i}\varepsilon_i \in R = A + \pi^{en}\bar{A}$, there are $a_i \in A$ and $\bar{a}_i \in \bar{A}$ such that $\pi^{n_i}\varepsilon_i = \pi^{n_i}a_i + \pi^{en}\bar{a}_i$ for $i = 1, 2$. Now we find

$$\begin{aligned}\pi^{en}\alpha &= (\pi^{n_1}\varepsilon_1)(\pi^{n_2}\varepsilon_2)\\ &= (\pi^{n_1}a_1 + \pi^{en}\bar{a}_1)(\pi^{n_2}a_2 + \pi^{en}\bar{a}_2)\\ &= \pi^{en}(a_1a_2 + \pi^{n_1}a_1\bar{a}_2 + \pi^{n_2}\bar{a}_1a_2 + \pi^{en}\bar{a}_1\bar{a}_2\,,\end{aligned}$$

and therefore

$$\alpha = a_1a_2 + \pi^{n_1}a_1\bar{a}_2 + \pi^{n_2}\bar{a}_1a_2 + \pi^{en}\bar{a}_1\bar{a}_2\,,$$

where $a_1a_2 \in A$, and thus $\pi^{en-\min\{n_1,n_2\}}\alpha \in R$. Since $en - \min\{n_1, n_2\} < en$, this is a contradiction to $\pi^l\alpha \notin R$ for $l < en$. Thus we have $\pi^{en}\alpha \in \mathcal{A}(R)$, and therefore $en = \mathsf{v}_p(\pi^{en}\alpha) \in \mathsf{v}_\pi(\mathcal{A}(R))$. $\square$

Definition 4.1.6. Let R be a Dedekind domain with quotient field K, L/K be a finite separable field extension, let $\bar{R}$ be the integral closure of R in L , and $f \in R^\bullet \setminus R^\times$. Then we set $\bar{R}_f = R + f\bar{R}$.

By the theorem of Krull-Akizuki then $\bar{R}$ is a Dedekind domain and since L/K is finite $\bar{R}$ is a finitely generated R module, i.e. there are $n \in \mathbb{N}$ and $r_2, \ldots, r_n \in \bar{R}$ such that $\bar{R} = R + r_2R + \ldots + r_nR$.
Obviously, $\bar{R}_f$ is an order in $\bar{R}$. Now we calculate its conductor. The inclusion $f\bar{R} \subset (\bar{R}_f : \bar{R})$ is clear. Let $x = a + fb \in (\bar{R}_f : \bar{R}) \subset \bar{R}_f = R + f\bar{R}$ with $a \in R$ and $b \in \bar{R}$. Then we find $a = x - fb \in (\bar{R}_f : \bar{R}) \cap R$, and thus $a\bar{R} \subset \bar{R}_f$. Now we have

$$ar_2 = g + f\sum_{i=2}^{n} l_ir_r \quad \text{with } g, l_2, \ldots, l_n \in R.$$

By comparison of coefficients we find $a = fl_2 \in fR$, and thus $x = fl_2 + fb = f(l_2 + b) \in f\bar{R}$. So we have $(\bar{R}_f : \bar{R}) = f\bar{R}$.

Let $\mathfrak{p} \in \mathfrak{X}(R)$ and $n \in \mathbb{N}$ with $f \in \mathfrak{p}^n$ and $f \notin \mathfrak{p}^{n+1}$. Then the localization $(\bar{R}_f)_\mathfrak{p} = R_\mathfrak{p} + \mathfrak{p}_\mathfrak{p}^n\bar{R}_\mathfrak{p}$ is an order in $\bar{R}_\mathfrak{p}$ with conductor $((\bar{R}_r)_\mathfrak{p} : \bar{R}_\mathfrak{p}) = \mathfrak{p}_\mathfrak{p}^n\bar{R}_\mathfrak{p}$.

In the following lemma we can construct the strict monoids we will make use of in Section 4.2.4.2.4.

Lemma 4.1.7. *Let R be a Dedekind domain with quotient field K, L/K be a finite separable field extension, let $\bar{R}$ be the integral closure of R in K, $f \in R^\bullet \setminus R^\times$, and $\bar{R}_f = R + f\bar{R}$.*

1. *Let $\mathfrak{p} \in \mathfrak{X}(R)$, $n \in \mathbb{N}$ such that $f \in \mathfrak{p}^n$ and $f \notin \mathfrak{p}^{n+1}$. Let $s \in \mathbb{N}$, $\pi_1, \ldots, \pi_s$ be a complete system of pairwise non-associated prime elements of $\bar{R}_\mathfrak{p}$, for $i \in [1, s]$, let $\mathsf{v}_{\pi_i} : L^\bullet \to \mathbb{Z}$ be the π_i-adic valuation of $L^\bullet$, and $\mathbf{e} = (\mathsf{v}_{\pi_1}(\mathfrak{p}), \ldots, \mathsf{v}_{\pi_s}(\mathfrak{p})) \in \mathbb{N}^s$. Then $(\bar{R}_f)_\mathfrak{p}^\bullet \subset \bar{R}_\mathfrak{p}^\bullet = [\pi_1, \ldots, \pi_s] \times \bar{R}_\mathfrak{p}^\times$ is a strict monoid of type $(\mathbf{e}, n)$.*
2. *Let $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\bar{R}_f) \mid f \in \mathfrak{p}\}$ and $\mathcal{P} = \mathfrak{X}(\bar{R}_f) \setminus \mathcal{P}^*$, $\mathrm{Pic}(\bar{R}_f)$ be finite, and let every class in $\mathrm{Pic}(\bar{R}_f)$ contain a prime $\mathfrak{p} \in \mathcal{P}$.*
Then
$$(\bar{R}_f^\bullet)_{\mathrm{red}} \subset \mathcal{F}(\mathcal{P}) \times T \cong \mathcal{I}^*(\bar{R}_f) \text{ with } T = \prod_{\mathfrak{p}\in\mathcal{P}^*} ((\bar{R}_f)_\mathfrak{p}^\bullet)_{\mathrm{red}}$$
is a saturated and cofinal submonoid with class group $\mathrm{Pic}(\bar{R}_f) = \mathcal{I}^*(\bar{R}_f)/(\bar{R}_f^\bullet)_{\mathrm{red}}$. Furthermore, for all $\mathfrak{p} \in \mathcal{P}^*$, $(\bar{R}_f)_\mathfrak{p}^\bullet \subset \bar{R}_\mathfrak{p}^\bullet$ is a strict monoid.

3. *Let $R = \mathbb{Z}$, $K = \mathbb{Q}$, L be an algebraic number field, $\bar{R} = \mathcal{O}_L$ be the integral closure of R in L and the maximal order in L, and $f \in \mathbb{Z}$.*
   Then $\bar{R}_f = \mathcal{O}_{L,f}$ fulfilled the assumptions of part 2.
4. *Let F be a field, $K = F(t)$, $R = F[t]$, L be an algebraic function field over F such that $\bar{R}$, the integral closure of R in L, is a finitely generated R-module, and $f(t) \in F[t]$.*
   Then $\bar{R}_{f(t)}$ fulfilled the assumptions of part 2.

Proof.

1. Obvious by Proposition 4.1.4.3.
2. The main assertion follows by [**14**, Theorem 3.7.1]. By part 1, the additional statement follows, since, obviously, a monoid is a strict monoid if and only if its associated reduced monoid is a strict monoid.
3. By [**14**, Corollary 2.11.16], $\mathrm{Pic}(\mathcal{O}_{L,f})$ is finite and each class in $\mathrm{Pic}(\mathcal{O}_{L,f})$ contains some $\mathfrak{p} \in \mathcal{P}$.
4. By [**14**, Proposition 8.9.7], $\mathrm{Pic}(R_{K,f(t)})$ is finite and each class contains some prime element $\mathfrak{p} \in \mathcal{P}$. □

4.2. $\min \triangle(R_f)$ and $\min \triangle(\mathcal{O}_{K,f})$

4.2.1. The split case.

Lemma 4.2.1. *Let H be an atomic monoid. If, for all $m \geq 2$, there exists $x_m \in H$ such that, for all $k \in [0, m-1]$, there are atoms $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(H)$ with $a_{m,k} b_{m,k} c_{m,k}^k = x_m$, then we have*

- $\min \triangle(H) = 1$*;*
- *for all $l \geq 2$, $\mathcal{V}_l(H) = [2, \infty)$; and*
- *for all finite subsets $I \subset \mathbb{N}_{\geq 2}$ there is $x \in H$ such that $I \subset \mathsf{L}(x)$.*

Proof. Let $m \in \mathbb{N}$ be arbitrary. By assumption, there is $x_m \in H$ such that, for all $k \in [0, m-1]$, there are atoms $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(H)$ with $a_{m,k} b_{m,k} c_{m,k}^k = x_m$. Thus we find $[2, m+1] \subset \mathsf{L}(x_m)$, and therefore $\min \triangle(H) = 1$. Since m was arbitrary, we have, for all $l \geq 2$,

$$[2, \infty) = \bigcup_{m \geq l+1} [2, m+1] \subset \bigcup_{m \geq l+1} \mathsf{L}(x_m) \subset \mathcal{V}_l(H) \subset [2, \infty)\,,$$

an thus equality holds. For all $I \subset \mathbb{N}_{\geq 2}$, we have $I \subset [2, \max(I)] \subset \mathsf{L}(x_{\max(I)-1})$. □

Lemma 4.2.2. *Let $H \subset \widehat{H} = [p_1, \ldots, p_s] \times \widehat{H}^\times$ be a strict monoid of type $(\mathbf{e}, n)$ of rank $s \geq 2$ with $n \in \mathbb{N}$.*
Then we have $p_1^{e_1 n + n_1} \cdot \ldots \cdot p_s^{e_s n + n_s} \varepsilon \in \mathcal{A}(H)$ for all $\varepsilon \in \widehat{H}^\times$ and $n_1, \ldots, n_s \in \mathbb{N}_0$ where $n_i \geq 1$ and $n_j < e_j$ for some i, $j \in [1, s]$.

Proof. Let $n_1 < e_1$ without loss of generality, and set $a = p_1^{e_1 n + n_1} \cdot \ldots \cdot p_s^{e_s n + n_s} \varepsilon$ with $\varepsilon \in \widehat{H}^\times$. Then $a \in H$, since $p_1^{e_1 n} \cdot \ldots \cdot p_s^{e_s n} \widehat{H} \subset H$. Suppose $a \notin \mathcal{A}(H)$. Then there are b, $c \in H \backslash H^\times$ with $a = bc$. Now we find $n e_i + n_i = \mathsf{v}_{p_i}(a) = \mathsf{v}_{p_i}(b) + \mathsf{v}_{p_i}(c)$ for $i \in [1, s]$. Since b, $c \notin H^\times$ and $n_1 < e_1$, we obtain $\mathsf{v}_{p_1}(b)$, $\mathsf{v}_{p_1}(c) \in e_1\mathbb{N} \cup \mathbb{N}_{\geq e_1 n}$ and $e_1 n > \mathsf{v}_{p_1}(b)$, $\mathsf{v}_{p_1}(c) \geq e_1$. Therefore, there are l_b, $l_c \in [1, n)$ such that $\mathsf{v}_{p_1}(b) = l_b e_1$ and $\mathsf{v}_{p_1}(c) = l_c e_1$. We have $n e_1 + n_1 = \mathsf{v}_{p_1}(a) = \mathsf{v}_{p_1}(b) + \mathsf{v}_{p_1}(c) = (l_b + l_c) e_1$, thus $n_1 = 0$ and

$l_b + l_c = n$. Let $i \in [2, n]$. Since H is a strict monoid and $n > l_b, l_c$, we find $\mathsf{v}_{p_i}(b) = l_b e_i$ and $\mathsf{v}_{p_i}(c) = l_c e_i$. Now we get $ne_i + n_i = \mathsf{v}_{p_i}(a) = \mathsf{v}_{p_i}(b) + \mathsf{v}_{p_i}(c) = ne_i$, and therefore $n_i = 0$. This is a contradiction, because, by assumption, we have $\max\{n_1, \ldots, n_s\} \geq 1$. $\square$

THEOREM 4.2.3. *Let D be a monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $D_1 \subset T$ be a divisor-closed submonoid and $D_1 \subset \widehat{D_1} = [p_1, \ldots, p_s] \times \widehat{D_1}^\times$ a strict monoid of type $(\mathbf{e}, n)$ of rank $s \geq 2$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ be its class group with $\#G \leq 2$, and let each class in G contain a prime element of P.*
Then we have

- $\min \triangle(H) = 1$;
- *for all $I \subset \mathbb{N}_{\geq 2}$ finite there is $x \in H$ such that $I \subset \mathsf{L}(x)$; and*
- *$\mathcal{V}_l(H) = [2, \infty)$ for all $l \geq 2$.*

PROOF. By Lemma 1.2.16, it suffices to show the assertion for the T-block monoid $\mathcal{B}(G, T, \iota)$ defined by the homomorphism $\iota : T \to G$, $\iota(t) = [t]_{D/H}$. For short, we write $\mathcal{B} = \mathcal{B}(G, T, \iota)$ and $p = p_2^{ne_2} \cdot \ldots \cdot p_s^{ne_s}$. Let $m \geq 2$ and $k \in [0, m-1]$.
First we study the case $\#G = 1$. Here we have $H = D$ and $\mathcal{B} = T$. We set

$$\begin{aligned} a_{m,k} &= p_1^{ne_1} p^{2m-k+1}, \\ b_{m,k} &= p_1^{(2m-2k+1)ne_1} p, \\ c_{m,k} &= p_1^{2ne_1} p, \end{aligned}$$

By Lemma 4.2.2, we find $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(D_1)$ and $a_{m,k} b_{m,k} c_{m,k}^k = x_m$ for $k \in [0, m-1]$ with

$$\begin{aligned} x_m &= a_{m,k} b_{m,k} c_{m,k}^k \\ &= (p_1^{ne_1} p^{2m-k+1})(p_1^{(2m-2k+1)ne_1} p)(p_1^{2ne_1} p)^k \\ &= p_1^{ne_1 + (2m-2k+1)ne_1 + (2ne_1)k} p^{2m-k+1+1+k} \\ &= (p_1^{ne_1} p)^{2m+2}. \end{aligned}$$

Since $D_1 \subset T$ is a divisor-closed submonoid, we have $\mathcal{A}(D_1) \subset \mathcal{A}(T) = \mathcal{A}(\mathcal{B})$.
Now let $\#G = 2$, say $G = \{\mathbf{0}, g\}$. Here we distinguish three cases.
If $\iota(p_1^{e_1 n}) = \iota(p) = \mathbf{0}$, then we set

$$\begin{aligned} a_{m,k} &= p_1^{e_1 n} p^{2m-k+1}, \\ b_{m,k} &= p_1^{e_1 n(2m-2k+1)} p, \\ c_{m,k} &= p_1^{2e_1 n} p. \end{aligned}$$

By Lemma 4.2.2, we find $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$ and, since $\iota(a_{m,k}) = \iota(b_{m,k}) = \iota(c_{m,k}) = \mathbf{0}$, we find $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(\mathcal{B})$.
If $\iota(p_1^{e_1 n}) = g$ and $\iota(p) = \mathbf{0}$, then we set

$$\begin{aligned} a'_{m,k} &= p_1^{e_1 n} p^{2m-k+1}, \\ b'_{m,k} &= p_1^{e_1 n(2m-2k+1)} p, \\ c_{m,k} &= p_1^{2e_1 n} p. \end{aligned}$$

$a_{m,k} = a'_{m,k}g$, and $b_{m,k} = b'_{m,k}g$. By Lemma 4.2.2, we have $a'_{m,k}$, $b'_{m,k}$, $c_{m,k} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$, and, since $\iota(a_{m,k}) = \iota(b_{m,k}) = \iota(c_{m,k}) = \mathbf{0}$, we have $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(\mathcal{B})$.
If $\iota(p_1^{e_1 n}) = \mathbf{0}$ and $\iota(p) = g$, then there is some $j \in [2, s]$ such that $\iota(p_j^{e_j n}) = g$ and $\iota(pp_1^{e_1 n}p_j^{-e_j n}) = \mathbf{0}$. Thus we can assume $\iota(p_1^{e_1 n}) = g$ and $\iota(p) = \mathbf{0}$ without loss of generality.
If $\iota(p_1^{e_1 n}) = \iota(p) = g$, then we set

$$\begin{aligned}
a'_{m,k} &= p_1^{e_1 n}p^{m-k+1},\\
b'_{m,k} &= p_1^{e_1 n(3m-3k)}p,\\
c_{m,k} &= p_1^{3e_1 n}p.
\end{aligned}$$

and

$$a_{m,k} = \begin{cases} a'_{m,k}g & k-m \text{ is odd,}\\ a'_{m,k} & k-m \text{ is even,}\end{cases}$$

$$b_{m,k} = \begin{cases} b'_{m,k}g & k-m \text{ is even,}\\ b'_{m,k} & k-m \text{ is odd.}\end{cases}$$

By Lemma 4.2.2, we have $a'_{m,k}$, $b'_{m,k}$, $c_{m,k} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$, and, since $\iota(a_{m,k}) = \iota(b_{m,k}) = \iota(c_{m,k}) = \mathbf{0}$, we have $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(\mathcal{B})$.
In all cases we found atoms $a_{m,k}$, $b_{m,k}$, $c_{m,k} \in \mathcal{A}(\mathcal{B})$ for all $m \geq 2$ and for all $k \in [0, m-1]$, which fulfill the conditions of Lemma 4.2.1. Thus the assertions follow. $\square$

4.2.2. The non-split cases.

THEOREM 4.2.4. *Let D be a monoid, $P \subset D$ be a set of prime elements, and $T \subset D$ be an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $D_1 \subset T$ be a divisor-closed submonoid and $D_1 \subset [p] \times \widehat{D_1}^{\times} = \widehat{D_1}$ be a monoid of type (e, en) of rank 1 with e, $n \in \mathbb{N}$. Let $H \subset D$ be a saturated atomic submonoid, $G = \mathsf{q}(D/H)$ be its class group with $\#G \leq 2$, and let each class in G contain some $p' \in P$.*
Suppose one of the following conditions is fulfilled.

1. *$e \geq 4$.*
2. *$e = 3$ and $p \in \widehat{H}$.*
3. *$e = 2$, $p \in \widehat{H}$, and there exists some $a \in \mathcal{A}(D_1)$ such that $\mathsf{v}_p(a) = 2n$.*
4. *$e = 1$, $n \geq 2$, $p \in \widehat{H} \cap D_1$, and there exist ε, $\eta \in \widehat{D_1}^{\times}$ such that $p^n\varepsilon$, $p^2\eta$, $p^n\varepsilon\eta \in \mathcal{A}(D_1)$, and $[\eta]_{D/H} = \mathbf{0}$.*
5. *$e = 1$, $n = 2$, $p \in \widehat{H} \cap D_1$, and there exists $\varepsilon \in \widehat{D_1}^{\times}$ such that $p^2\varepsilon$, $p^2\varepsilon^2 \in \mathcal{A}(D_1)$.*

Then we have

$$\min\triangle(H) = 1.$$

PROOF. By Lemma 1.2.16, it suffices to show the assertion for the T-block monoid $\mathcal{B}(G, T, \iota)$ defined by the homomorphism $\iota : T \to G$, $\iota(t) = [t]_{D/H}$. For short, we write $\mathcal{B} = \mathcal{B}(G, T, \iota)$.

1. Since $e \geq 4$, we have p^{en+1}, p^{en+2}, $p^{en+3} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$.
   Now suppose $\iota(p) = \mathbf{0}$. Then $\iota(p^{en+1}) = \iota(p^{en+2}) = \mathbf{0}$, hence p^{en+1}, $p^{en+2} \in \mathcal{A}(\mathcal{B})$, and the assertion follows from

$$(p^{en+1})^{en+2} = (p^{en+2})^{en+1}.$$

Now suppose $\iota(p) = g$. We notice that $e \geq 5$ if e is odd and that in this case $p^{en+4} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$. Now we set

$$i = \begin{cases} en+1 & e \text{ is even,} \\ en+2 & e \text{ is odd.} \end{cases}$$

Then p^i, $p^{i+2} \in \mathcal{A}(D_1)$ and $\iota(p^i) = \iota(p^{i+2}) = g$ in all cases, and therefore $p^i g$, $p^{i+2}g$, $g^2 \in \mathcal{A}(\mathcal{B})$. Now the assertion follows from

$$(p^i g)^{i+2} = (p^{i+2}g)^i g^2.$$

2. Since $e = 3$, we have p^{en+1}, $p^{en+2} \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$. Furthermore, $\iota(p^{en+1}) = \iota(p^{en+2}) = \mathbf{0}$, hence p^{en+1}, $p^{en+2} \in \mathcal{A}(\mathcal{B})$, and the assertion follows from

$$(p^{en+1})^{en+2} = (p^{en+2})^{en+1}.$$

3. Let $a \in \mathcal{A}(D_1)$ be an atom of D_1 such that $\mathsf{v}_p(a) = 2n$. Then there is $\varepsilon \in \widehat{D_1}^\times$ such that $a = p^{4n}\varepsilon$. We have $p^{4n}\varepsilon$, p^{4n+1}, $p^{4n+1}\varepsilon \in \mathcal{A}(D_1)$. Now we consider two cases.
If $\iota(\varepsilon) = \mathbf{0}$, we have $\iota(p^{4n}\varepsilon) = \iota(p^{4n+1}) = \iota(p^{4n+1}\varepsilon) = \mathbf{0}$, and hence $p^{4n}\varepsilon$, p^{4n+1}, $p^{4n+1}\varepsilon \in \mathcal{A}(\mathcal{B})$. Now the assertion follows from

$$(p^{4n}\varepsilon)^{4n+1}p^{4n+1} = (p^{4n+1}\varepsilon)^{4n+1}.$$

If $\iota(\varepsilon) = g$, we have $\iota(p^{4n}\varepsilon g) = \iota(p^{4n+1}) = \iota(p^{4n+1}\varepsilon g) = \mathbf{0}$, and hence $p^{4n}\varepsilon g$, p^{4n+1}, $p^{4n+1}\varepsilon g \in \mathcal{A}(\mathcal{B})$. Now the assertion follows from

$$(p^{4n}\varepsilon g)^{4n+1}p^{4n+1} = (p^{4n+1}\varepsilon g)^{4n+1}.$$

4. By assumption we have $\iota(\eta) = \mathbf{0}$ and p, $p^n\varepsilon$, $p^2\eta$, $p^n\varepsilon\eta \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$. Now we consider two cases.
If $\iota(\varepsilon) = \mathbf{0}$, then $\iota(p^n\varepsilon) = \iota(p^2\eta) = \iota(p^n\varepsilon\eta) = \mathbf{0}$, and therefore p, $p^n\varepsilon$, $p^2\eta$, $p^n\varepsilon\eta \in \mathcal{A}(\mathcal{B})$. Thus the assertion follows from

$$(p^n\varepsilon)(p^2\eta) = p^2(p^n\varepsilon\eta).$$

If $\iota(\varepsilon) = g$, then $\iota(p^n\varepsilon g) = \iota(p^2\eta) = \iota(p^n\varepsilon\eta g) = \mathbf{0}$, and therefore p, $p^n\varepsilon$, $p^2\eta$, $p^n\varepsilon\eta g \in \mathcal{A}(\mathcal{B})$. Thus the assertion follows from

$$(p^n\varepsilon g)(p^2\eta) = p^2(p^n\varepsilon\eta g).$$

5. By assumption we have p, $p^2\varepsilon$, $p^2\varepsilon^2 \in \mathcal{A}(D_1) \subset \mathcal{A}(T)$ and $\iota(\varepsilon) = \mathbf{0}$. Thus $\iota(p^2\varepsilon) = \iota(p^2\varepsilon^2) = \mathbf{0}$ and we find p, $p^2\varepsilon$, $p^2\varepsilon^2 \in \mathcal{A}(\mathcal{B})$. Now the assertion follows from

$$p^2(p^2\varepsilon^2) = (p^2\varepsilon)^2. \qquad \square$$

4.2.3. $\min \triangle((\mathcal{O}_{K,f})_{(p)})$ for local quadratic inert orders.

This subsection heavily relies on the calculations in [**17**]. First, we fix some notations.

4.2.3.1. *Notations.* Let $d \in \mathbb{Z}\backslash\{0,1\}$ be square-free. Then $K = \mathbb{Q}(\sqrt{d})$ is a quadratic number field. Let $p \in \mathbb{P}$ be a prime, such that p is inert with respect to K. Set

$$\omega = \begin{cases} \frac{3+\sqrt{d}}{4} & \text{if } d \equiv 1 \mod 4 \text{ and } p = 2, \\ \sqrt{d} & \text{else.} \end{cases}$$

Then $\mathbb{Z}_{(p)}[\omega] = (\mathcal{O}_K)_{(p)}$ is the integral closure of $\mathbb{Z}_{(p)}$ in K and at the same time it is the localization of K's ring of integers $\mathcal{O}_K$.
Now let $n \in \mathbb{N}$. $(\mathcal{O}_{K,p^n})_{(p)}$ is the local order or equivalently the $\mathbb{Z}_{(p)}$-order in $(\mathcal{O}_K)_{(p)}$ with conductor $((\mathcal{O}_{K,p^n})_{(p)} : (\mathcal{O}_K)_{(p)}) = p^n(\mathcal{O}_K)_{(p)}$. Then $(\mathcal{O}_{K,p^n})_{(p)} = \mathbb{Z}_{(p)}[p^n\omega] = \mathbb{Z}_{(p)}[\tau]$ with

$$\tau = \begin{cases} 2^{n-1}\sqrt{d} & \text{if } d \equiv 1 \mod 4 \text{ and } p = 2, \\ \sqrt{d} & \text{else.} \end{cases}$$

Since p is inert in K, we have $\left(\frac{d}{p}\right) = -1$.
We now fix n and p and we write $\bar{R} = (\mathcal{O}_K)_{(p)}$ and $R = (\mathcal{O}_{K,p^n})_{(p)}$ for short.
For $m \in \mathbb{N}$ and $c \in R$, we write $u_{m,c} = p^m c + \tau$ and for $M \in \mathbb{N}$ we set $\Omega(M) = \{c \in [0, p^m - 1] \mid c + \tau \in R\backslash(R^\times \cup pR), \mathcal{N}(c+\tau) \sim p^m\}$.

Remark 4.2.5. If we look at a local order $(\mathcal{O}_{K,f})_{(p)}$ with $f \in \mathbb{N}_{\geq 2}$, then we have $(\mathcal{O}_{K,f})_{(p)} = (\mathcal{O}_{K,p^n})_{(p)}$ where $n \in \mathbb{N}$ such that $p^n \mid f$ and $p^{n+1} \nmid f$. Thus we will write $(\mathcal{O}_{K,p^n})_{(p)}$ instead of $(\mathcal{O}_{K,f})_{(p)}$ form now on.

4.2.3.2. *The monoid $R^\bullet$.*
By Lemma 4.1.7.3 $R^\bullet \subset \bar{R}^\bullet$ is a strict monoid of type $(1,n)$ of rank n. Thus it is clear that $\rho(R) \leq n$ and it is a well known fact that $\rho(R) = 1$ if and only if $n = 1$.

4.2.3.3. *The two possible cases.*

- $p \neq 2$
- $p = 2$

4.2.3.4. *Case $p \neq 2$.*
This case is discussed in detail in [**17**]. Here we give only a brief summary of the results.
Complete system of pairwise not-associated atoms

type (a)	p
type (b)	$u_{m,c}$ with $m \in [1, n-1]$, $c \in [0, p^m - 1]$, $p \nmid c$
type (c)	$u_{n,c}$ with $c \in [1, p^n - 1]$

Let $u_{n,c}$ be an atom of type (c) and set $p^k c_1 = c$ with $k \in \mathbb{N}_0$. Then we have $u_{n,c} = u_{n+k,c_1}$. Since $c \in [1, 2^n - 1]$, we find $k \in [0, n-1]$ and $c_1 \in [1, p^{n-k} - 1]$. Each $k \in [0, n-1]$ appears.
Relations among these atoms

Lemma 4.2.6. *Let $j \in [1, n-1]$ and $c \in [0, p^{n-j} - 1]$ with $p \nmid c$. Let $c_0 \in \mathbb{Z}$ be such that $p \nmid c_0$ and*

$$\left(\frac{d - cc_0}{p}\right) = 1.$$

Then there exists some $c_1 \in [0, p^{n-j} - 1]$ such that

$$c_1^2 - p^j c_0 c_1 \equiv d - cc_0 \mod p^{n-j}.$$

If c_1 is chosen in this way and if $c_2 \in [0, p^{n-j} - 1]$ satisfies $c_2 \equiv p^j c_0 - c_1 \mod p^{n-j}$, then choose any $c_3 \in [0, p^n - 1]$ with $p \nmid c_3(c_3 - c_1)$ and let $c_4 \in [0, p^n - 1]$ be such that

$$c_4 \equiv \frac{c_1 c_3 - d}{c_3 - c_1} \mod p^n.$$

Then we have

$$p^{4n+j} u_{n-j,c} \sim u_{n,c_2} u_{n,c_3} u_{n,c_4}.$$

PROOF. By [**17**, Lemma 0.5], it follows that

$$u_{n,c_3} u_{n,c_4} \sim p^n u_{n,c_1}$$

and, by [**17**, Lemma 0.6], we find

$$u_{n,c_1} u_{n,c_2} \sim p^{n+j} u_{n-j,c}.$$

Now we can combine those results to obtain

$$u_{n,c_2} u_{n,c_3} u_{n,c_4} \sim p^n u_{n,c_1} u_{n,c_2} \sim p^{4n+j} u_{n-j,c}. \qquad \square$$

Unions of sets of lengths

By [**17**, Lemma 0.7.2], there is $x \in R$ such that $[2, 3] \in \mathsf{L}(x)$. By [**17**, Lemma 0.6], we have $m + j + 1 \subset \mathcal{V}_2(R)$ for all $m \in [1, n]$ and $j \in [1, m - 1]$. Thus we have $[4, 2n] \subset \mathcal{V}_2(R)$. Now we find

$$[2, 2n] = [2, 3] \cup [4, 2n] \subset \mathcal{V}_2(R) \subset \left[\max\left\{2, \left\lceil\frac{4}{n}\right\rceil\right\}, 2n\right] = [2, 2n]$$

Since $[2, 2n] = \mathcal{V}_2(R)$, we have $[2, 2n + 1] \subset \mathcal{V}_3(R)$. By Lemma 4.2.6, we have $2n + j + 1 \in \mathcal{V}_3(R)$ for all $j \in [1, n - 1]$, and therefore $[2n + 2, 3n] \subset \mathcal{V}_3(R)$. Now we find

$$[2, 3n] \subset \mathcal{V}_3(R) \subset \left[\max\left\{2, \left\lceil\frac{5}{n}\right\rceil\right\}, 3n\right] = [2, 3n].$$

4.2.3.5. *Case $p = 2$.*

Since 2 is inert, i.e. $\left(\frac{d}{2}\right) = -1$, we have $d \equiv 5 \mod 8$, thus $d \equiv 1 \mod 4$.

The sets $\Omega(M)$

By [**17**, Lemma 0.4 B], we have

$$\Omega(M) = \begin{cases} \emptyset & M < 2n \text{ and } M \equiv 1 \mod 2, \\ \{2^m c_0 | c_0 \in [0, 2^m - 1],\ 2 \nmid c_0\} & M + 2m \text{ with } m \in [1, n-2], \\ \{2^n c_1 | c_1 \in [0, 2^{n-2} - 1]\} & M = 2n - 2, \\ \{2^{n-1} c_0 | c_0 \in [0, 2^{n+1} - 1]\ 2 \nmid c_0\} & M = 2n, \\ \emptyset & M > 2n. \end{cases}$$

Complete system of pairwise not-associated atoms

type (a)	2,
type (b)	$u_{m,c}$ with $m \in [1, n-2]$, $c \in [2^m - 1]$, $2 \nmid c$,
type (c)	$u_{n-1,c}$ with $c \in [0, 2^{n+1} - 1]$, $2 \nmid c$,
type (d)	$u_{n,c}$ with $c \in [0, 2^{n-2} - 1]$.

Let $u_{n,c}$ be an atom of type (d) and set $p^k c_1 = c$ for $k \in \mathbb{N}_0$. Then we have $u_{n,c} = u_{n+k,c_1}$. Since $c \in [1, 2^{n-2} - 1]$, we find $k \in [0, n - 3]$ and $c_1 \in [1, p^{n-k-2} - 1]$. Each $k \in [0, n - 3]$ appears.

Relations among these atoms
More or less the same as in the case $p \neq 2$.
Unions of sets of lengths

$$\mathcal{V}_2(R) = [2, 2n] \quad \text{and} \quad \mathcal{V}_3(R) = [2.3n].$$

4.2.3.6. *Unions of sets of lengths (all cases).*

LEMMA 4.2.7. *Let H be an atomic monoid and $e \in \mathbb{N}$ minimal such that $e\rho(H) \in \mathbb{N}$. If, for all $k \in [2, \max\{3, e+1\}]$,*

$$\mathcal{V}_k(H) = \left[\max\left\{2, \left\lceil \frac{k}{\rho(H)} \right\rceil\right\}, \lfloor k\rho(H) \rfloor\right] \tag{4.2.1}$$

is satisfied, then (4.2.1) *holds for all $k \geq 2$.*

PROOF. For short, we set $n = \rho(H)$. Suppose now $e = 1$. In this case, we prove the assertion by induction on k. Let $k \geq 4$ and suppose the assumption is true for all $2 \leq k' < k$. Let l, $l' \in \mathbb{N}_0$ be such that $k = 2l + 3l'$. Then we have

$$[2(l+l'), kn] = [2(l+l'), (2l+3l')n] = l\mathcal{V}_2(H) + l'\mathcal{V}_3(H) \subset \mathcal{V}_k(H).$$

It remains to show that

$$\left[\max\left\{2, \left\lceil \frac{k}{n} \right\rceil\right\}, 2(l+l') - 1\right] \subset \mathcal{V}_k(H).$$

Let $k' \in \left[\max\left\{2, \left\lceil \frac{k}{n} \right\rceil\right\}, 2(l+l') - 1\right]$. Then we find

$$\frac{k}{n} \leq k' \leq 2(l+l') - 1 < 2(l+l')' \leq 2l + 3l' = k,$$

and therefore $k'n \geq k > k'$. By induction hypotheses, it follows that $k \in \mathcal{V}_{k'}(H)$, and therefore $k' \in \mathcal{V}_k(H)$.
Now suppose $e > 1$. Then we have $\max\{2, e+1\} = e+1$ and we prove the assertion again by induction on k. Let $k \geq e+2$ and suppose the assertion is proven for $k' < k$. k has a unique decomposition of the form $k = ek' + k''$ with $k' \in \mathbb{N}$ and $k'' \in [2, e+1]$. We have $k'\mathcal{V}_e(H) + \mathcal{V}_{k''}(H) \subset \mathcal{V}_k(H)$. Since $en \in \mathbb{N}$, we have $\lfloor kn \rfloor = \lfloor (k'e + k'')n \rfloor = k'en + \lfloor k''n \rfloor$, and therefore $[k, \lfloor kn \rfloor] = [k'e, e'en] + [k'', \lfloor k'', \lfloor k'n \rfloor] \subset \mathcal{V}_{k'e}(H) + \mathcal{V}_{k''}(H) \subset \mathcal{V}_k(H)$. It remains to show that

$$M = \left[\max\left\{2, \left\lceil \frac{k}{n} \right\rceil\right\}, k\right) \subset \mathcal{V}_k(H).$$

Let $k' \in M$. Then

$$\frac{k}{n} \leq k' < k,$$

and therefore $k' < k \leq k'n$. Since $k \in \mathbb{N}$, we have $k \leq \lfloor k'n \rfloor$, and therefore we have $k \in \mathcal{V}_{k'}(H)$, i.e. $k' \in \mathcal{V}_k(H)$ by induction hypothesis. □

COROLLARY 4.2.8. *Let $k \in \mathbb{N}_{\geq 2}$. Then*

$$\mathcal{V}_k(R) = \left[\max\left\{2, \left\lceil \frac{k}{n} \right\rceil\right\}, kn\right].$$

PROOF. Follows from $\mathcal{V}_2(R) = [2, 2n]$ and $\mathcal{V}_3(R) = [2, 3n]$ by Lemma 4.2.7 since $\rho(R) = n \in \mathbb{N}$. □

4.2.4. The theorems on $\min \triangle(R_f)$ and on $\min \triangle(\mathcal{O}_{K,f})$.

THEOREM 4.2.9. *Let R be a Dedekind domain with quotient field K, L/K be a finite separable field extension, let $\bar{R} = \mathsf{cl}_K(R)$, $f \in R^\bullet \setminus R^\times$, $\bar{R}_f = R + f\bar{R}$, let each class in* $\mathrm{Pic}(\bar{R}_f)$ *contain a prime, and set $\mathcal{P}^* = \{\mathfrak{p} \in \mathfrak{X}(\bar{R}_f) \mid f \in \mathfrak{p}\}$.*

1. *If some $\mathfrak{p} \in \mathcal{P}^*$ splits in $\bar{R}$, then:*
   - *For all finite subsets $I \subset \mathbb{N}_{\geq 2}$ there is some $a \in \mathcal{I}^*(\bar{R}_f)$ such that $I \subset \mathsf{L}(a)$.*
     In particular, $\mathcal{V}_l(\mathcal{I}^*(\bar{R}_f)) = [2, \infty)$ for all $l \geq 2$.
   - *If, additionally, $\# \mathrm{Pic}(\bar{R}_f) \leq 2$, then, for all finite subsets $I \subset \mathbb{N}_{\geq 2}$, there is some $a \in \bar{R}_f$ such that $I \subset \mathsf{L}(a)$.*
     In particular, $\mathcal{V}_l(\bar{R}_f) = [2, \infty)$ for all $l \geq 2$.
2. *If one of the conditions*
   (a) $\# \mathrm{Pic}(\bar{R}_f) \geq 3$,
   (b) some $\mathfrak{p} \in \mathcal{P}^*$ splits in $\bar{R}$, or
   (c) some $\mathfrak{p} \in \mathcal{P}^*$ has ramification index ≥ 4 in $\bar{R}$
   holds, then $\min \triangle(\bar{R}_f) = 1$.
   If (b) or (c) holds, then we have additionally $\min \triangle(\mathcal{I}^*(\bar{R}_f)) = 1$.

In particular, we can choose $R = \mathbb{Z}$, $K = \mathbb{Q}$, L an algebraic number field, $\bar{R} = \mathcal{O}_L = \mathrm{cl}_L(R)$, and $f \in \mathbb{Z}^\bullet \setminus \mathbb{Z}^\times$ or $R = F[t]$ with F a field, $K = F(t)$, L be an algebraic function field over F such that $\bar{R} = \mathrm{cl}_L(R)$ is a finitely generated R-module, and $f(t) \in F[t]^\bullet \setminus F^\times$.

THEOREM 4.2.10. *Let K be an algebraic number field, $f \in \mathbb{N}_{\geq 2}$, $\mathcal{P}^* = \{p \in \mathbb{P} \mid p \mid f\}$.*

1. *If some $p \in \mathcal{P}^*$ splits in $\mathcal{O}_K$, then:*
   - *For all finite subsets $I \subset \mathbb{N}_{\geq 2}$, there is some $a \in \mathcal{I}^*(\mathcal{O}_{K,f})$ such that $I \subset \mathsf{L}(a)$.*
     In particular, $\mathcal{V}_l(\mathcal{I}^*(\mathcal{O}_{K,f})) = [2, \infty)$ for all $l \geq 2$.
   - *If, additionally, $\# \mathrm{Pic}(\mathcal{O}_{K,f}) \leq 2$, then, for all finite subsets $I \subset N_{\geq 2}$, there is some $a \in \mathcal{O}_{K,f}$ such that $I \subset \mathsf{L}(a)$.*
     In particular, $\mathcal{V}_l(\mathcal{O}_{K,f}) = [2, \infty)$ for all $l \geq 2$.
2. *If either*
   (a) $\# \mathrm{Pic}(\mathcal{O}_{K,f}) \geq 3$,
   (b) there is $p \in \mathcal{P}^*$ which splits in $\mathcal{O}_K$,
   (c) there is $p \in \mathcal{P}^*$ which has ramification index ≥ 4 in $\mathcal{O}_K$,
   (d) there is $p \in \mathcal{P}^*$ and $\mathfrak{p} \in \mathrm{spec}(\mathcal{O}_K)$ with $\mathfrak{p} \cap \mathcal{O}_{K,f} = p\mathcal{O}_{K,f}$ such that $\mathfrak{p}$ is a principal ideal (in $\mathcal{O}_K$) and $(\mathcal{O}_{K,f})_{(p)}$ is not half-factorial, or
   (e) $\mathcal{O}_{K,f}$ is locally half-factorial,
   then $\min \triangle(\mathcal{O}_{K,f}) \leq 1$.
   If (b),(c), or (e) holds, then we have additionally $\min \triangle(\mathcal{I}^*(\mathcal{O}_{K,f})) \leq 1$.

In particular, if (e) holds, then $\min \triangle(\mathcal{I}^*(\mathcal{O}_{K,f})) = 0$.

PROOF OF THEOREM 4.2.9. In the whole proof we use the results from Lemma 4.1.7.2 without any further reference. The additional result follows immediately by Lemma 4.1.7.3 and by Lemma 4.1.7.4.

1. Let $\mathfrak{p} \in \mathcal{P}^*$ split in $\bar{R}$. Now we apply Theorem 4.2.3 with $H = D = \mathcal{I}^*(\bar{R}_f)$, therefore $\#\mathsf{q}(D/H) = 1$, and $D_1 = ((\bar{R}_f)^\bullet_{(p)})_{\mathrm{red}}$ to obtain the assertion for $\mathcal{I}^*(\bar{R}_f)$.

If $\#\operatorname{Pic}(\bar{R}_f) \le 2$, we can apply Theorem 4.2.3 a second time with $H = (\bar{R}_f^\bullet)_{\mathrm{red}}$ to obtain the assertion for $\bar{R}_f$.

2. (a) Let $\#\operatorname{Pic}(\bar{R}_f) \ge 3$. If we apply Lemma 1.2.17.1 with $H = \bar{R}_f^\bullet \subset D = \mathcal{I}^*(\bar{R}_f)$, we find $\min\triangle(\bar{R}_f) = 1$.
   (b) In this particular case the assertion follows from part 1.
   (c) Let $\mathfrak{p} \in \mathcal{P}^*$ be a prime which does not split in $\bar{R}$ (otherwise everything is proven in (b)) and which has ramification index ≥ 4 in $\bar{R}$. We can apply Theorem 4.2.4.1 with $H = D = \mathcal{I}^*(\bar{R}_f)$ and $D_1 = ((\bar{R}_f)_{(p)}^\bullet)_{\mathrm{red}}$. Then we get $\min\triangle(\mathcal{I}^*(\bar{R}_f)) = 1$. If $\#\operatorname{Pic}(\bar{R}_f) \ge 3$, we have $\min\triangle(\bar{R}_f) = 1$ by (a). So now we can assume $\#\operatorname{Pic}(\bar{R}_f) \le 2$. Hence, we can apply Theorem 4.2.4.1 a second time with $H = (\bar{R}_f^\bullet)_{\mathrm{red}}$ and we find $\min\triangle(\bar{R}_f) = 1$. □

PROOF OF THEOREM 4.2.10.

1. Follows from Theorem 4.2.9.1.
2. We show $\min\triangle(\mathcal{I}^*(\mathcal{O}_{K,f})) \le 1$ and $\min\triangle(\mathcal{O}_{K.f}) \le 1$ for each of the conditions (a)-(e). In all cases but (a) we assume $\#\operatorname{Pic}(\mathcal{O}_{K,f}) \le 2$. The cases (a)-(c) follow from the corresponding cases in Theorem 4.2.9.2. So now we only have to deal with (d) and (e).
   (d) Let $p \in \mathcal{P}^*$ be a prime such that there is $\mathfrak{p} \in \operatorname{spec}(\mathcal{O}_K)$ with $\mathfrak{p} \in \mathcal{O}_{K,f} = p\mathcal{O}_{K,f}$ that is a principal ideal (in $\mathcal{O}_K$). Therefore $[\mathfrak{p}]_{\operatorname{Pic}(\mathcal{O}_{K,f})} = [\mathfrak{p}]_{\mathcal{O}_K/\mathcal{O}_{K,f}} = \mathbf{0}$. If p splits in $\mathcal{O}_K$ we are in the situation of (b) and if p has ramification index ≥ 4 we are in the situation of (c). Therefore we may without loss of generality assume that p does not split in $\mathcal{O}_K$ and that p has ramification index $e \in \{1,2,3\}$. Now we do the proof case by case.
   If $e = 3$, then we can apply Theorem 4.2.4.2 with $D = \mathcal{I}^*(\mathcal{O}_{K,f})$, $D_1 = ((\mathcal{O}_{K,f})_{(p)}^\bullet)_{\mathrm{red}}$ and $H = \mathcal{I}^*(\mathcal{O}_{K,f})$ respectively $H = (\mathcal{O}_{K,f}^\bullet)_{\mathrm{red}}$. Then we get $\min\triangle(\mathcal{I}^*(\mathcal{O}_{K,f})) \le 1$ respectively $\min\triangle(\mathcal{O}_{K,f}) \le 1$.
   If $e = 2$, then let $n \in \mathbb{N}$ be such that $p^n \mid f$ and $p^{n+1} \nmid f$. Let $\mathfrak{p} \in \mathfrak{X}(\mathcal{O}_K)$ be a prime ideal such that $\mathfrak{p} \cap \mathcal{O}_{K,f} = p\mathcal{O}_{K,f}$ and $\mathfrak{p} = \bar{p}\mathcal{O}_K$. By Lemma 4.1.5 we find an atom $a \in \mathcal{A}(((\mathcal{O}_{K,f})_{(p)}^\bullet)_{\mathrm{red}})$ such that $\mathsf{v}_{\bar{p}}(a) = 2n$. Now the assertion follows by Theorem 4.2.4.3 with $D = \mathcal{I}^*(\mathcal{O}_{K,f})$, $D_1 = ((\mathcal{O}_{K,f})_{(p)}^\bullet)_{\mathrm{red}}$, $H = \mathcal{I}^*(\mathcal{O}_{K,f})$, and $H = \mathcal{O}_{K,f}^\bullet$ for $\mathcal{I}^*(\mathcal{O}_{K,f})$ respectively for $\mathcal{O}_{K,f}$.
   If $e = 1$, then we set $n \in \mathbb{N}$ such that $p^n \mid f$ and $p^{n+1} \nmid f$. By Proposition 4.1.4.2 $(\mathcal{O}_{K,f})_{(p)}^\bullet$ is a monoid of type $(1,n)$. If $n = 1$, then $(\mathcal{O}_{K,f})_{(p)}$ is half-factorial – a contradiction. Thus we can assume $n \ge 2$.
   First we deal with $n \ge 3$. By Lemma 4.1.5 there is $a \in \mathcal{A}(((\mathcal{O}_{K,f})_{(p)}^\bullet)_{\mathrm{red}})$ such that $a = p^n\alpha$ with $\alpha \in (((\mathcal{O}_K)_{(p)})_{\mathrm{red}})^\times$ We set H, D, and D_1 as before. Now we can apply Theorem 4.2.4.4 with $\varepsilon = \alpha$ and $\eta = 1 + p^{n-2}\alpha$ if $[\eta]_{\operatorname{Pic}(\mathcal{O}_{K,f})} = \mathbf{0}$. Otherwise we can set $\eta = (1 + p^{n-2}\alpha)^2$.
   Now let $n = 2$ and $k = [K : \mathbb{Q}]$. For all $\alpha \in (\mathcal{O}_K)_{(p)}^\times \setminus (\mathbb{Z}_{(p)}) + p(\mathcal{O}_K)_{(p)}^\times) = (\mathcal{O}_{K,p})_{(p)}^\times$ we have $p^2\alpha \in \mathcal{A}((\mathcal{O}_{K,f})_{(p)})$ (This follows immediately from the proof of Lemma 4.1.5.).
   A relation of type

$$p^2\alpha^2 = p(p\alpha^2) \quad \text{or} \quad (p^2\alpha^2) = (p\beta)(p\gamma)$$

with $\beta, \gamma \in (\mathcal{O}_{K,p})_{(p)}^{\times}$ is equivalent to $\bar{\alpha}^2 \in \mathbb{F}_p^{\times}$ ($^{-} : (\mathcal{O}_K)_{(p)} \to \mathbb{F}_{p^k}$ is the canonical mapping). But this cannot be the case for all elements since

$$\#\mathbb{F}_{p^k}^{\times} = p^k - 1 > 2(p-1) = 2\#\mathbb{F}_p^{\times}$$

for all $p \in \mathbb{P}$ and $k \geq 2$. Thus there is $\alpha \in (\mathcal{O}_K)_{(p)}^{\times}$ such that $p^2\alpha$, $p^2\alpha^2 \in \mathcal{A}((\mathcal{O}_{K,f})_{(p)})$. Now we can apply Theorem 4.2.4.5 if $[\alpha]_{\mathrm{Pic}(\mathcal{O}_{K,f})} = \mathbf{0}$.
If $[\alpha]_{\mathrm{Pic}(\mathcal{O}_{K,f})} = g$, then we consider α^2 and α^4 instead. For $k \geq 3$, the inequality

$$\#\mathbb{F}_{p^k}^{\times} = p^k - 1 > 4(p-1) = 4\#\mathbb{F}_p^{\times}$$

holds for all $p \in \mathbb{P}$. Thus the assertion follows again by Theorem 4.2.4.5.
If $k = 2$, we reuse the explicit calculations from subsection 4.2.3. The atoms in [**17**, Lemma 0.5] can be chosen globally. Thus the assertion follows.

(e) In this particular case the assertion follows from Corollary 3.1.21.2. □

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