

A CHARACTERIZATION OF ARITHMETICAL INVARIANTS BY THE MONOID OF RELATIONS II: THE MONOTONE CATENARY DEGREE AND APPLICATIONS TO SEMIGROUP RINGS

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ABSTRACT. The investigation and classification of non-unique factorization phenomena has attracted some interest in recent literature. For finitely generated monoids, S.T. Chapman and P.A. García-Sánchez, together with several co-authors, derived a method to calculate the catenary and tame degree from the monoid of relations. Then, in [24], the algebraic structure of this approach was investigated and the restriction to finitely generated monoids was removed. We now extend these ideas further to the monotone catenary degree and then apply all these results to the explicit computation of arithmetical invariants of semigroup rings.

1. INTRODUCTION

An integral domain and, more generally, a commutative, cancellative monoid is called *atomic* if every non-zero non-unit has a factorization into irreducible elements, and it is called *factorial* if this factorization is unique up to ordering and associates. Non-unique factorization theory is concerned with the description and classification of non-unique factorization phenomena arising in atomic domains. It has its origin in algebraic number theory—the ring of integers of an algebraic number field being atomic but generally not factorial—but in the last decades it became an autonomous theory with many connections to zero-sum theory, commutative ring theory, module theory, additive combinatorics, and representations of monoids. We refer to the monograph [14] for a recent presentation of the various aspects of the theory.

To describe these phenomena, various invariants have been studied in the literature. Among these, the tame degree, the catenary degree, and—a variant thereof—the monotone catenary degree received some attention in recent research; for some new results, see, e.g. [3], [12], and [13]; for an overview of known results and additional references, see, e.g., the monograph [14]; for a statement of the formal definitions, see Section 2 and the beginning of Section 4. Additionally, monotone and near monotone chains of factorizations have been studied in [10], [11], and [12, Section 7].

For an integral domain, non-unique factorization phenomena only concern the multiplicative monoid of that domain. Thus we will only derive the theory for commutative, cancellative monoids, and apply these results again to domains afterwards.

The monoid of relations associated to a monoid and a certain invariant $\mu(\cdot)$ have been used successfully to study the catenary degree and other invariants. Investigations of this type started only fairly recently. In [29], such investigations were carried out for finitely generated monoids using results from [4] and [5]. In [6] and [22], these results and expansions thereof were applied in the investigation of numerical monoids; for a detailed exposition of numerical monoids and applications, see, e.g. the monograph [28]. In [24] the algebraic structure of this method was studied: i.e., the invariant $\mu(\cdot)$, its definition, and the monoid of relations. By this more algebraic-structural approach, the results could be extended to not necessarily finitely generated monoids.

In the present paper, we extend the tools from [24] to study the monotone catenary degree by submonoids of the monoid of relations. In Section 5, we apply these results and many of the results from [24] to the explicit computation of arithmetical invariants for various semigroup rings. Additionally, the arithmetic of some generalized power series rings is studied there.

Moreover, these abstract characterizations, and, in particular, Lemma 3.4 are used successfully for investigations on the arithmetic of non-principal orders in algebraic number fields in [26].

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2. PRELIMINARIES

In this note, our notation and terminology will be consistent with [14]. Let \mathbb{N} denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \uplus \{0\}$. For integers $n, m \in \mathbb{Z}$, we set $[n, m] = \{x \in \mathbb{Z} \mid n \leq x \leq m\}$. By convention, the supremum of the empty set is zero and we set $\frac{0}{0} = 1$. The term “monoid” always means a commutative, cancellative semigroup with unit element. We will write all monoids multiplicatively. For a monoid H , we denote by H^\times the set of invertible elements of H . We call H reduced if $H^\times = \{1\}$ and call $H_{\text{red}} = H/H^\times$ the reduced monoid associated with H . Of course, H_{red} is always reduced. Note that the arithmetic of H is determined by H_{red} and therefore we can restrict our attention to reduced monoids whenever convenient. We denote by $\mathcal{A}(H)$ the set of atoms of H , by $\mathcal{A}(H_{\text{red}})$ the set of atoms of the associated reduced monoid H_{red} , by $Z(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$ the free (abelian) monoid with basis $\mathcal{A}(H_{\text{red}})$, and by $\pi_H : Z(H) \rightarrow H_{\text{red}}$ the unique homomorphism such that $\pi_H|_{\mathcal{A}(H_{\text{red}})} = \text{id}$. We call $Z(H)$ the factorization monoid and π_H the factorization homomorphism of H . For $a \in H$, we denote by $Z(a) = \pi_H^{-1}(aH^\times)$ the set of factorizations of a and denote by $L(a) = \{|z| \mid z \in Z(a)\}$ the set of lengths of a , where $|\cdot|$ is the ordinary length function in the free monoid $Z(H)$. In this terminology, a monoid H is called *half-factorial* if $|L(a)| = 1$ for all $a \in H \setminus H^\times$ —this coincides with the classical definition of being half-factorial, since then every two factorizations of an element have the same length—and *factorial* if $|Z(a)| = 1$ for all $a \in H \setminus H^\times$.

With all these notions at hand, for $a \in H$, we set

$$\rho(a) = \frac{\sup L(a)}{\min L(a)} \quad \text{and call } \rho(H) = \sup\{\rho(a) \mid a \in H\} \text{ the elasticity of } H.$$

Note that H is half-factorial if and only if $\rho(H) = 1$.

For two factorizations $z, z' \in Z(H)$, we call

$$d(z, z') = \max \left\{ \left| \frac{z}{\gcd(z, z')} \right|, \left| \frac{z'}{\gcd(z, z')} \right| \right\} \quad \text{the distance between } z \text{ and } z'$$

and, for two subset $X, Y \subset Z(H)$, we call

$$d(X, Y) = \min\{d(x, y) \mid x \in X, y \in Y\} \quad \text{the distance between } X \text{ and } Y.$$

If one of the sets is a singleton, say $X = \{x\}$, we write $d(\{x\}, Y) = d(x, Y)$.

Let $a \in H$. We call two lengths $k, l \in L(a)$ with $k < l$ *adjacent* if $[k, l] \cap L(a) = \{k, l\}$ and, for $M \subset \mathbb{N}$, we set $Z_M(a) = \{x \in Z(a) \mid |x| \in M\}$. If the set is a singleton, say $M = \{k\}$, then we write $Z_{\{k\}}(a) = Z_k(a)$.

Definition 2.1. Let H be an atomic monoid and let $a \in H$.

1. Factorizations $z_0, \dots, z_n \in Z(a)$ with $n \in \mathbb{N}$ and $d(z_{i-1}, z_i) \leq N$ for some $N \in \mathbb{N}$ and $i \in [1, n]$ are called

- an N -chain concatenating z_0 and z_n (in $Z(H)$).
- a *monotone* N -chain concatenating z_0 and z_n (in $Z(H)$) if $|z_{i-1}| \leq |z_i|$ for all $i \in [1, n]$.
- an *equal-length* N -chain concatenating z_0 and z_n (in $Z(H)$) if $|z_{i-1}| = |z_i|$ for all $i \in [1, n]$.

2. The

- *catenary degree* $c(a)$
- *monotone catenary degree* $c_{\text{mon}}(a)$
- *equal catenary degree* $c_{\text{eq}}(a)$

denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for all $z, z' \in Z(a)$, there is

- an N -chain concatenating z and z' .
- a monotone N -chain concatenating z and z' .
- an equal-length N -chain concatenating z and z' .

Then we call

- $c(H) = \sup\{c(a) \mid a \in H\}$ the *catenary degree* of H .
- $c_{\text{mon}}(H) = \sup\{c_{\text{mon}}(a) \mid a \in H\}$ the *monotone catenary degree* of H .
- $c_{\text{eq}}(H) = \sup\{c_{\text{eq}}(a) \mid a \in H\}$ the *equal-length catenary degree* of H .

Note that $\sup\{c(H), c_{\text{eq}}(H)\} \leq c_{\text{mon}}(H)$.

For the description and computation of the monotone catenary degree, we follow the same two step procedure as in [3]. In order to formulate this precisely, we need to define another variant of the catenary degree.

Definition 2.2. Let H be an atomic monoid. For $a \in H$, we define

$$c_{\text{ad}}(a) = \sup\{d(Z_k(a), Z_l(a)) \mid k, l \in L(a) \text{ are adjacent}\}$$

as the *adjacent catenary degree* of a .

Also, $c_{\text{ad}}(H) = \sup\{c_{\text{ad}}(a) \mid a \in H\}$ is called the *adjacent catenary degree* of H .

By [3, (4.1)], we find

$$c(H) \leq c_{\text{mon}}(H) = \sup\{c_{\text{eq}}(H), c_{\text{ad}}(H)\}.$$

Here we follow the same strategy as in [24, Section 3] for the definition of the \mathcal{R} -relation and the μ -invariant.

Definition 2.3. Let H be an atomic monoid and let $a \in H$.

1. Factorizations $z_0, \dots, z_n \in Z(a)$ with $n \in \mathbb{N}$ and $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, n]$ are called
 - an \mathcal{R} -chain concatenating z_0 and z_n (in $Z(H)$).
 - a *monotone* \mathcal{R} -chain concatenating z_0 and z_n (in $Z(H)$) if $|z_{i-1}| \leq |z_i|$ for all $i \in [1, n]$.
 - an *equal-length* \mathcal{R} -chain concatenating z_0 and z_n (in $Z(H)$) if $|z_{i-1}| = |z_i|$ for all $i \in [1, n]$.
2. Two elements $z, z' \in Z(H)$ are
 - \mathcal{R} -related
 - \mathcal{R}_{eq} -related
 if there is an
 - \mathcal{R} -chain
 - equal-length \mathcal{R} -chain
 concatenating z and z' . We then write $z \approx z'$ respectively $z \approx_{\text{eq}} z'$.

Note that, with the above definitions, \approx and \approx_{eq} are congruences on $Z(H) \times Z(H)$.

Based on these definitions, we can now recall the definition of the μ -invariant (for reference see [24, Section 3]) and give the definition of the μ_{eq} -invariant and the μ_{ad} -invariant. Note that the definition of the last one differs significantly from the other two since there is no appropriate equivalence relation we can make use of.

Definition 2.4. Let H be an atomic monoid and let $a \in H$.

1. Let \mathcal{R}_a denote the set of \mathcal{R} -equivalence classes of $Z(a)$ and, for $\rho \in \mathcal{R}_a$, let $|\rho| = \min\{|z| \mid z \in \rho\}$. We set

$$\mu(a) = \sup\{|\rho| \mid \rho \in \mathcal{R}_a\} \leq \sup L(a)$$

and define $\mu(H) = \sup\{\mu(a) \mid a \in H\}$.

2. For $k \in L(a)$, let $\mathcal{R}_{a,k}$ denote the set of \mathcal{R}_{eq} -equivalence classes of $Z_k(a)$. We set

$$\mu_{\text{eq}}(a) = \sup\{k \in L(a) \mid |\mathcal{R}_{a,k}| > 1\} \leq \sup L(a)$$

and define $\mu_{\text{eq}}(H) = \sup\{\mu_{\text{eq}}(a) \mid a \in H\}$.

3. We set

$$\mu_{\text{ad}}(a) = \sup\{k \in L(a) \mid d(Z_k(a), Z_l(a)) = k \text{ for } l \in L(a), l < k, l \text{ adjacent to } k\}.$$

Then we set $\mu_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H\}$.

Then $\mu(H) = 0$ if and only if $|\mathcal{R}_a| \leq 1$ for all $a \in H$ and $\mu_{\text{eq}}(H) = 0$ if and only if $|\mathcal{R}_{a,k}| \leq 1$ for all $a \in H$ and $k \in L(a)$.

Definition 2.5. Let $H \subset D$ be monoids.

1. We call $H \subset D$ *saturated* or, equivalently, a *saturated submonoid* if, for all $a, b \in H$, $a \mid b$ in D already implies that $a \mid b$ in H ; that is, for all $a, b \in H$ and $c \in D$, $a = bc$ implies $c \in H$.
2. If $H \subset D$ is a saturated submonoid, then we set $D/H = \{a\mathbf{q}(H) \mid a \in D\}$ and $[a]_{D/H} = a\mathbf{q}(H)$ and we call $\mathbf{q}(D)/\mathbf{q}(H) = \mathbf{q}(D/H)$ the *class group* of H in D .

Definition 2.6. Let H be an atomic monoid. We call

$$\sim_H = \{(x, y) \in Z(H) \times Z(H) \mid \pi(x) = \pi(y)\} \quad \text{the monoid of relations of } H,$$

$$\sim_{H,\text{eq}} = \{(x, y) \in \sim_H \mid |x| = |y|\} \quad \text{the monoid of equal-length relations of } H,$$

$$\sim_{H,\text{mon}} = \{(x, y) \in \sim_H \mid |x| \leq |y|\} \quad \text{the monoid of monotone relations of } H,$$

and, for $a \in H$, we set

$$\begin{aligned} \mathcal{A}_a(\sim_H) &= \mathcal{A}(\sim_H) \cap (Z(a) \times Z(a)), \\ \mathcal{A}_a(\sim_{H,\text{eq}}) &= \mathcal{A}(\sim_{H,\text{eq}}) \cap (Z(a) \times Z(a)), \\ \mathcal{A}_a(\sim_{H,\text{mon}}) &= \mathcal{A}(\sim_{H,\text{mon}}) \cap (Z(a) \times Z(a)). \end{aligned}$$

By [24, Lemma 11], $\sim_H \subset Z(H) \times Z(H)$ is a saturated submonoid of a free monoid and thus a Krull monoid by [14, Theorem 2.4.8.1]. By [3, Proposition 4.4.1], $\sim_{H,\text{eq}} \subset \sim_H$ is a saturated submonoid and hence a Krull monoid, and, by [3, Proposition 4.4.2], $\sim_{H,\text{eq}}$ is finitely generated if H_{red} is finitely generated. Unfortunately, $\sim_{H,\text{mon}} \subset \sim_H$ is not saturated, but, by Lemma 5.11, we find that $\sim_{H,\text{mon}}$ is a finitely generated Krull monoid if H is finitely generated.

We briefly recall the main result on the catenary degree from [24] and offer a corrected proof for monoids fulfilling the ascending chain condition on principal ideals here.

Lemma 2.7 (cf. [24, Proposition 8, Corollary 9, Proposition 16]). *Let H be an atomic monoid which fulfills the ascending chain condition on principal ideals. Then*

1. $c(H) \geq \mu(a)$ for all $a \in H$, and $c(H) = \mu(H)$.
2. $c(H) = \max\{\mu(a) \mid a \in H, \mathcal{A}_a(\sim_H) \neq \emptyset, |\mathcal{R}_a| > 1\}$.

Proof.

1. First we prove

$$c(a) \geq \mu(a) \text{ for all } a \in H.$$

Let $a \in H$ be such that $|\mathcal{R}_a| > 1$. We may assume that $c(a) < \infty$. Let $N \in \mathbb{N}_0$ be such that $\mu(a) \geq N$. Let $\rho \in \mathcal{R}_a$ be such that $|\rho| \geq N$ and $z \in \rho$ such that $|z| = |\rho|$. Let $z' \in Z(a)$ be such that $z \not\approx z'$ and let $z = z_0, z_1, \dots, z_k = z'$ be a $c(a)$ -chain concatenating z and z' . Let $i \in [1, k]$ be minimal such that $z \not\approx z_i$. Then $z_{i-1} \not\approx z_i$, and therefore

$$N \leq |z_0| \leq |z_{i-1}| \leq d(z_i, z_{i-1}) \leq c(a).$$

Till now we have $c(H) \geq \mu(H)$. Next we show

$$\mu(H) \geq c(H).$$

We show that, for all $a \in H$, we have $c(a) \leq \mu(H)$. We proceed by induction on a . For $a = 1$, this is trivial. Now suppose $a \neq 1$ and that, for all $b \in H$ with $b \mid a$, we have $c(b) \leq \mu(H)$. Now let $z, z' \in Z(a)$. If $z \approx z'$, then there are $z'', z''' \in Z(a)$ such that $z'' \approx z, z''' \approx z'$, and z'' and z''' are minimal in their \mathcal{R} -classes with respect to their lengths. Since $\gcd(z'', z''') = 1$, we find $d(z'', z''') = \max\{|z''|, |z'''|\} \leq \mu(a) \leq \mu(H)$. Now it remains to show that, for any two factorizations $z, z' \in Z(a)$ with $z \approx z'$, there is a $\mu(H)$ -chain concatenating them. By definition, there is an \mathcal{R} -chain z_0, \dots, z_k with $z = z_0$ and $z' = z_k$, and $g_i = \gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. Since $\pi_H(g_i^{-1} z_{i-1}) \mid a$, we find a $\mu(H)$ -chain concatenating $g_i^{-1} z_{i-1}$ and $g_i^{-1} z_i$ for all $i \in [1, k]$ by induction hypothesis, and thus there is a $\mu(H)$ -chain concatenating z_{i-1} and z_i for all $i \in [1, k]$; thus there is a $\mu(H)$ -chain concatenating z and z' . So $c(a) \leq \mu(H)$.

2. When we compare the definitions, we see that the only thing remaining is

$$\{a \in H \mid \mathcal{A}_a(\sim_H) \neq \emptyset, |\mathcal{R}_a| > 1\} = \{a \in H \mid |\mathcal{R}_a| > 1\}.$$

One inclusion is trivial and, for the other one, let $a \in H$ be such that $|\mathcal{R}_a| > 1$, and let $z, z' \in Z(a)$ be two factorizations of a such that $z \not\approx z'$ and such that both are minimal in their \mathcal{R} -equivalence classes with respect to their lengths. Now assume $(z, z') \notin \mathcal{A}(\sim_H)$. Then there is $k \geq 2$ and $(x_1, y_1), \dots, (x_k, y_k) \in \mathcal{A}(\sim_H)$ such that $(z, z') = (x_1, y_1) \dots (x_k, y_k)$. But now we find the following \mathcal{R} -chain from z to z' : $z_0 = z$ and $z_i = z_{i-1} x_i^{-1} y_i$ for $i \in [1, k]$. Then $z_k = z'$ and $\gcd(z_{i-1}, z_i) \neq 1$. Since this is a contradiction, we have $(z, z') \in \mathcal{A}(\sim_H)$, and thus $(z, z') \in \mathcal{A}_a(\sim_H) \neq \emptyset$. \square

3. A CHARACTERIZATION OF THE MONOTONE CATENARY DEGREE BY MONOIDS OF RELATIONS

Lemma 3.1. *Let H be an atomic monoid, $a \in H$ and $x, y \in Z(a)$.*

1. *If $x \not\approx_{\text{eq}} y$, then $(x, y) \in \mathcal{A}_a(\sim_{H,\text{eq}})$.*
2. *Let $k, l \in \mathbb{L}(a)$ be adjacent with $k < l$. If $d(Z_k(a), Z_l(a)) = l$, then $(x, y) \in \mathcal{A}_a(\sim_{H,\text{mon}})$ for all $x \in Z_k(a)$ and $y \in Z_l(a)$.*

Proof. Since the arithmetic of H is determined solely by H_{red} , we may assume that H is reduced.

1. Let $a \in H$ and $x, y \in Z(a)$ be such that $(x, y) \notin \mathcal{A}_a(\sim_{H,\text{eq}})$. Then, trivially, $(x, y) \notin \mathcal{A}(\sim_{H,\text{eq}})$ and thus there are $(x_1, y_1), \dots, (x_k, y_k) \in \mathcal{A}(\sim_{H,\text{eq}})$ with $k \geq 2$ such that $(x, y) = (x_1, y_1) \dots (x_k, y_k)$. Then $x = x_1 \dots x_k, y_1 x_2 \dots x_k, y_1 \dots y_k = y$ is an \mathcal{R}_{eq} -chain concatenating x and y , and therefore $x \approx_{\text{eq}} y$.

2. Let $a \in H$, let $k, l \in \mathbf{L}(a)$ be adjacent with $k < l$ and $\mathbf{d}(Z_k(a), Z_l(a)) = l$, and let $x \in Z_k(a)$ and $y \in Z_l(a)$. Now suppose $(x, y) \notin \mathcal{A}_a(\sim_{H, \text{mon}})$. Then, trivially, $(x, y) \notin \mathcal{A}(\sim_{H, \text{mon}})$ and there are $(x_1, y_1), \dots, (x_k, y_k) \in \mathcal{A}(\sim_{H, \text{mon}})$ with $k \geq 2$ and $|y_1| - |x_1| \leq \dots \leq |y_k| - |x_k|$. Then we set $x' = x_1^{-1}y_1x$. If $|y_1| - |x_1| = 0$, we find $|x'| = k$ and $\gcd(x', y) \neq 1$, a contradiction to $\mathbf{d}(Z_k(a), Z_l(a)) = l$. Otherwise, if $|y_1| - |x_1| > 0$, then $k = |x| < |x'| < |y| = l$, a contradiction to k and l being adjacent. \square

In principal, we follow the same strategy as in [24, Section 3] for the μ -invariant when studying the μ_{eq} -invariant. For the μ_{ad} -invariant, we cannot construct an equivalence relation like the \mathcal{R} -relation or the \mathcal{R}_{eq} -relation. Thus we follow a slightly modified strategy in the proofs of parts 3 and 4 from Theorem 3.2.

Theorem 3.2. *Let H be an atomic monoid. Then*

1. $\mathbf{c}_{\text{eq}}(a) \geq \mu_{\text{eq}}(a)$ for all $a \in H$, and $\mathbf{c}_{\text{eq}}(H) = \mu_{\text{eq}}(H)$.
2. $\mathbf{c}_{\text{eq}}(H) = \sup\{\mu_{\text{eq}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H, \text{eq}}) \neq \emptyset, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathbf{L}(a)\}$
 $= \sup\{k \in \mathbb{N} \mid a \in H, \mathcal{A}_a(\sim_{H, \text{eq}}) \neq \emptyset, k \in \mathbf{L}(a), |\mathcal{R}_{a,k}| > 1\}$.
3. $\mathbf{c}_{\text{ad}}(a) \geq \mu_{\text{ad}}(a)$ for all $a \in H$, and $\mathbf{c}_{\text{ad}}(H) = \mu_{\text{ad}}(H)$.
4. $\mathbf{c}_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H, \text{mon}}) \neq \emptyset\}$.

In particular,

$$\mathbf{c}_{\text{mon}}(H) = \sup(\{\mu_{\text{eq}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H, \text{eq}}), |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathbf{L}(a)\} \cup \{\mu_{\text{ad}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H, \text{mon}})\}).$$

Proof. Since the arithmetic of H is determined solely by H_{red} we may assume that H is reduced.

1. First we prove $\mathbf{c}_{\text{eq}}(a) \geq \mu_{\text{eq}}(a)$ for all $a \in H$. We may assume that $\mathbf{c}_{\text{eq}}(a) < \infty$ and $\mu_{\text{eq}}(a) \geq 1$. Let $N \in \mathbb{N}$ be such that $N \leq \mu_{\text{eq}}(a)$. Then there exists $k \in \mathbf{L}(a)$ such that $|\mathcal{R}_{a,k}| > 1$ and $k \geq N$. Let $z, z' \in Z_k(a)$ be such that $z \not\sim_{\text{eq}} z'$, and let $z = z_0, z_1, \dots, z_n = z'$ be a $\mathbf{c}_{\text{eq}}(a)$ -equal-length chain concatenating z and z' . Now we choose $i \in [0, n-1]$ minimal such that $z \not\sim_{\text{eq}} z_i$. Then $z_{i-1} \not\sim_{\text{eq}} z_i$, and we find

$$\mathbf{c}_{\text{eq}}(a) \geq \mathbf{d}(z_{i-1}, z_i) = k \geq N.$$

Now we prove $\mu_{\text{eq}}(H) \geq \mathbf{c}_{\text{eq}}(H)$. We show that, for all $N \in \mathbb{N}_0$, all $a \in H$, and all factorizations $z, z' \in Z(a)$ with $|z| = |z'| \leq N$, there is a $\mu_{\text{eq}}(H)$ -equal-length-chain from z to z' . We proceed by induction on N . If $N = 0$, then $z = z' = 1$ and $\mathbf{d}(z, z') = 0 \leq \mu_{\text{eq}}(H)$. Suppose $N \geq 1$ and that, for all $a \in H$ and all $z, z' \in Z(a)$ with $|z| = |z'| < N$, there is a $\mu_{\text{eq}}(H)$ -equal-length-chain from z to z' . Now let $a \in H$ and let $z, z' \in Z(a)$ with $|z| = |z'| \leq N$. If $z \sim_{\text{eq}} z'$, then $\mu_{\text{eq}}(H) \geq \mu_{\text{eq}}(a) \geq |z| = \mathbf{d}(z, z')$. Now it remains to show that, for any two factorizations $z, z' \in Z(a)$ with $|z| = |z'| \leq N$ and $z \not\sim_{\text{eq}} z'$, there is a $\mu_{\text{eq}}(H)$ -equal-length-chain concatenating them. By definition, there is an \mathcal{R}_{eq} -chain z_0, \dots, z_k with $z_0 = z, z' = z_k, g_i = \gcd(z_{i-1}, z_i) \neq 1$, and $|z_i| = |z|$ for all $i \in [1, k]$. By induction hypothesis, there is a $\mu_{\text{eq}}(H)$ -equal-length-chain from $g_i^{-1}z_{i-1}$ to $g_i^{-1}z_i$ for all $i \in [1, k]$, and hence there is a $\mu_{\text{eq}}(H)$ -equal-length-chain from z_{i-1} to z_i for $i \in [1, k]$; thus there is a $\mu_{\text{eq}}(H)$ -equal-length chain from z to z' .

2. By part 1, we have $\mathbf{c}_{\text{eq}}(H) = \mu_{\text{eq}}(H)$ and, by Definition 2.4.2, the third equality is obvious. Thus it suffices to show that

$$\{\mu_{\text{eq}}(a) \mid a \in H, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathbf{L}(a)\} = \{\mu_{\text{eq}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H, \text{eq}}) \neq \emptyset, |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathbf{L}(a)\}.$$

The inclusion from right to left is clear. Now let $a \in H$ and $k \in \mathbf{L}(a)$ be such that $|\mathcal{R}_{a,k}| > 1$. Then there exist $z, z' \in Z_k(a)$ such that $z \not\sim_{\text{eq}} z'$. By Lemma 3.1.1, we find $(z, z') \in \mathcal{A}_a(\sim_{H, \text{eq}}) \neq \emptyset$.

3. First let $a \in H$. We show that $\mathbf{c}_{\text{ad}}(a) \geq \mu_{\text{ad}}(a)$, and then $\mathbf{c}_{\text{ad}}(H) \geq \mu_{\text{ad}}(H)$ follows by passing to the supremum on both sides. If $\mu_{\text{ad}}(a) = 0$ or $\mu_{\text{ad}}(a) = \infty$, this is trivial. Now let $\mu_{\text{ad}}(a) = l \in \mathbb{N}$. Then there is $k \in \mathbf{L}(a)$ and $k < l$ with l adjacent to k . Then, by Definition 2.2, $\mathbf{c}_{\text{ad}}(a) \geq \mathbf{d}(Z_k(a), Z_l(a)) = \max\{k, l\} = l = \mu_{\text{ad}}(a)$.

Now we prove $\mu_{\text{ad}}(H) \geq \mathbf{c}_{\text{ad}}(H)$. We must prove that $\mathbf{c}_{\text{ad}}(a) \leq \mu_{\text{ad}}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathbf{c}_{\text{ad}}(a) > \mu_{\text{ad}}(H)$. Let $l \in \mathbb{N}$ be minimal such that there is some $k < l$ and $a \in H$ with k and l adjacent lengths of a and $\mathbf{c}_{\text{ad}}(a) = \mathbf{d}(Z_k(a), Z_l(a))$. If $\mathbf{d}(Z_k(a), Z_l(a)) < l$, then there are some $x \in Z_k(a)$ and $y \in Z_l(a)$ such that $g = \gcd(x, y) \neq 1$. If $b = \pi_H(g^{-1}x)$, then $k - |g|$ and $l - |g|$ are adjacent lengths of b and

$$\mathbf{c}_{\text{ad}}(a) = \mathbf{d}(Z_k(a), Z_l(a)) \leq \mathbf{d}(Z_{k-|g|}(b), Z_{l-|g|}(b)) \leq \mathbf{c}_{\text{ad}}(b),$$

and by the minimal choice of l , we infer that $\mathbf{c}_{\text{ad}}(b) \leq \mu_{\text{ad}}(H)$, a contradiction.

4. By part 3 and Definition 2.4.3, we find

$$c_{\text{ad}}(H) = \mu_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H\}.$$

Thus it suffices to show that

$$\sup\{\mu_{\text{ad}}(a) \mid a \in H\} = \sup\{\mu_{\text{ad}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H,\text{mon}})\}.$$

In fact, we only have to show that $\sup\{\mu_{\text{ad}}(a) \mid a \in H\} \leq \sup\{\mu_{\text{ad}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H,\text{mon}})\}$. Now let $a \in H$ and $\mu_{\text{ad}}(a) = k \in \mathbb{N}$. Then there is $l \in \mathbb{L}(a)$ with $l < k$, l adjacent to k , and $d(Z_k(a), Z_l(a)) = k$. Now let $x \in Z_l(a)$ and $y \in Z_k(a)$. Then we have $\gcd(x, y) = 1$. By Lemma 3.1.2, we have $(x, y) \in \mathcal{A}_a(\sim_{H,\text{mon}}) \neq \emptyset$.

The additional statement now follows easily by parts 2 and 4. \square

Lemma 3.3. *Let H be an atomic monoid and let $a \in H$.*

1. *Let $x, y \in Z(a)$ with $\min\{|x|, |y|\} > c_{\text{mon}}(a)$. Then there is a monotone \mathcal{R} -chain concatenating x and y , and thus $x \approx y$; in particular, if $|x| = |y|$, then $x \approx_{\text{eq}} y$.*
2. *Let $k, l \in \mathbb{L}(a)$. Then*

$$d(Z_k(a), Z_l(a)) = \max\{k, l\} \quad \text{if and only if} \quad \gcd(x, y) = 1 \text{ for all } x \in Z_k(a) \text{ and } y \in Z_l(a).$$
3. *Let $k, l \in \mathbb{L}(a)$ be adjacent with $k < l$ such that there are $x \in Z_k(a)$ and $y \in Z_l(a)$ such that there is a monotone \mathcal{R} -chain concatenating x and y . Then $\mu_{\text{ad}}(a) \neq 1$.*

Proof. Since the arithmetic of H is determined solely by H_{red} , we may assume that H is reduced.

1. Let $a \in H$ and $x, y \in Z(a)$ be such that $\min\{|x|, |y|\} > c_{\text{mon}}(a)$. We may assume that $|x| \leq |y|$. Then there is a monotone $c_{\text{mon}}(a)$ -chain concatenating x and y , say $z_0 = x, z_1, \dots, z_k = y$. Since, for all $i \in [1, k]$, we have $d(z_{i-1}, z_i) \leq c_{\text{mon}}(a) < |x| = |z_0|$, we have $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. Thus z_0, \dots, z_k is a monotone \mathcal{R} -chain concatenating x and y , and therefore $x \approx y$. If $|x| = |y|$, then z_0, \dots, z_k is an equal-length chain, and therefore $x \approx_{\text{eq}} y$.
2. Follows immediately by the definition of the distance of factorizations in $Z(H)$.
3. Let $a \in H$, let $k, l \in \mathbb{L}(a)$ be adjacent with $k < l$, let $x \in Z_k(a)$, and $y \in Z_l(a)$ be such that there is a monotone \mathcal{R} -chain from x to y , say $z_0 = x, z_1, \dots, z_n = y$ for some $n \in \mathbb{N}$. Now choose $i \in [1, n]$ minimal such that $|z_i| = l$. Due to the minimality of i , we find $z_{i-1} \in Z_k(a)$. Since $\gcd(z_{i-1}, z_i) \neq 1$, we find $d(Z_k(a), Z_l(a)) < l$, and therefore $\mu_{\text{ad}}(a) \neq 1$. \square

Lemma 3.4. *Let H be an atomic monoid. Then*

1. $c_{\text{eq}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\text{eq}}), x \not\approx_{\text{eq}} y\}$.
2. $c_{\text{ad}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\text{mon}}), |x| < |y|, |x|, |y| \in \mathbb{L}(\pi_H(x)) \text{ adjacent, and there is no monotone } \mathcal{R}\text{-chain from } x \text{ to } y\}$.

In particular,

$$c_{\text{mon}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\text{mon}}), \text{ there is no monotone } \mathcal{R}\text{-chain from } x \text{ to } y, \text{ and either } |x| = |y| \text{ or } |x|, |y| \in \mathbb{L}(\pi_H(x)) \text{ are adjacent}\}.$$

Proof. Since the arithmetic of H is determined solely by H_{red} , we may assume that H is reduced.

1. The inequality $c_{\text{eq}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\text{eq}})\}$ has been proven in [3, Proposition 4.4.3]. The slightly stronger statement here follows immediately by the definition of $\mu_{\text{eq}}(\cdot)$; see Definition 2.4.2.
2. By Theorem 3.2.4, we have $c_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H, \mathcal{A}_a(\sim_{H,\text{mon}}) \neq \emptyset\}$. Now the assertion follows from Lemma 3.3.3, Lemma 3.1.2, and the definition of $\mu_{\text{ad}}(\cdot)$; see Definition 2.4.3.

The additional statement follows from

$$c_{\text{mon}}(H) = \sup\{c_{\text{eq}}(H), c_{\text{ad}}(H)\} \quad \text{and} \quad \mathcal{A}(\sim_{H,\text{eq}}) \subset \mathcal{A}(\sim_{H,\text{mon}}). \quad \square$$

4. TAMENESS AND MONOTONE CHAINS

Definition 4.1. Let H be an atomic monoid.

1. For $a \in H$ and $x \in Z(H)$, let $t(a, x)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: If $Z(a) \cap xZ(H) \neq \emptyset$ and $z \in Z(a)$, then there exists some $z' \in Z(a) \cap xZ(H)$ such that $d(z, z') \leq N$.

For subsets $H' \subset H$ and $X \subset Z(H)$, we define

$$t(H', X) = \sup\{t(a, x) \mid a \in H', x \in X\},$$

and we define $t(H) = t(H, \mathcal{A}(H_{\text{red}}))$. This is called the *tame degree* of H .

2. If $t(H) < \infty$, then we call H *tame*.

Being tame is a very strong finiteness condition within non-unique factorization theory, in particular, the finiteness of the tame degree implies the finiteness of the elasticity and the catenary degree among other invariants. Next, we give a list of examples where tameness is characterized in various classes of monoids and domains; for a similar list, the reader is referred to [16, Examples 3.2].

1. *Finitely generated monoids*. If H_{red} is finitely generated, then H is tame (see [14, Theorem 3.1.4]).
2. *Finitely primary monoids*. Let H be finitely primary of rank $s \in \mathbb{N}$. Then H is tame if and only if $s = 1$ (see [14, Theorem 3.1.5]).
3. *Weakly Krull domains*. Let R be a v -noetherian weakly Krull domain with nonzero conductor $\mathfrak{f} = (R : \widehat{R})$ and finite v -class group $C_v(R)$. Note that, in particular, orders in algebraic number fields fulfill all these properties.
Then R is tame if and only if, for every nonzero prime ideal $\mathfrak{p} \in \mathfrak{X}(R)$ with $\mathfrak{p} \supset \mathfrak{f}$, there is precisely one $\mathfrak{P} \in \mathfrak{X}(\widehat{R})$ such that $\mathfrak{P} \cap R = \mathfrak{p}$ (see [14, Theorem 3.7.1]).
4. *Krull monoids and therefore Krull domains*. Let H be a Krull monoid, $F = \mathcal{F}(P)$ a monoid of divisors and $G_P = \{[p] \mid p \in P\} \subset F/H_{\text{red}} = G$ the set of classes containing prime divisors. Suppose that one of the following conditions hold:
 - (a) H has the approximation property.
 - (b) Every $g \in G_P$ contains at least two prime divisors.
 - (c) There is an $m \in \mathbb{N}$ such that $-G_P \subset m(G_P \cup \{0\})$.
 - (d) The torsion free rank of G is finite.

Then H is tame if and only if $D(G_P) < \infty$ (see [16, Theorem 4.2]). In particular, all principal orders in algebraic number fields are tame.

5. *C-like monoids*. Let H be a C -like monoid. Then H is tame if and only if the natural map $s\text{-spec}(\widehat{H}) \rightarrow s\text{-spec}(H)$ is bijective (see [19, Theorem 8.3] and [19, Definition 5.6] for a precise definition of C -like monoids).

Next we give two examples of C -like monoids. R^\bullet is a C -like monoid if

- (see [19, Proposition 6.1]) R is an integral domain and R^\bullet is finitely primary.
- (see [19, Proposition 6.5]) R is a Mori domain with complete integral closure \widehat{R} , $C_v(\widehat{R})$ is finite, $(R : \widehat{R}) \neq 0$, and either
 - R is semilocal, and $\widehat{R}/(R : \widehat{R})$ is quasi artinian or
 - $C_v(R)$ is finite and $S^{-1}\widehat{R}/S^{-1}(R : \widehat{R})$ is quasi artinian, where $S \subset R^\bullet$ is the submonoid of regular elements.

While the tameness of a monoid implies the finiteness of the catenary degree, it does not imply the finiteness of the equal catenary degree and therefore not the finiteness of the monotone catenary degree. In order to point this out, we discuss a monoid originally introduced in [9, Example 4.5].

Recall that a monoid H is called *finitely primary* if there exist $s, k \in \mathbb{N}$ and a factorial monoid $F = [p_1, \dots, p_s] \times F^\times$ with the following properties:

- $H \setminus H^\times \subset p_1 \cdot \dots \cdot p_s F$ and
- $(p_1 \cdot \dots \cdot p_s)^k F \subset H$.

If this is the case, then we call H a *finitely primary monoid of rank s and exponent k* .

Example 4.2 (cf. [9, Example 4.5]). There exists a tame monoid H such that $c_{\text{eq}}(H) = \infty$, and thus $c_{\text{mon}}(H) = \infty$ but $c_{\text{ad}}(H) < \infty$.

Proof. We proceed in four steps.

1. We start with a construction which was first used in [18]. Let G be an additively written abelian group and $f : G \rightarrow \mathbb{N}_0$ a map with $f(0) = 0$ and finite image $f(G)$ such that, for all $g, g' \in G$, the following two conditions are satisfied:
 - (a) $f(g + g') \leq f(g) + f(g')$ and
 - (b) if $f(g) = 0$, then $f(-g) = 0$.

Then, by construction,

$$H(G, f) = \{(g, k) \mid g \in G, k \in \mathbb{N}_0 \text{ with } k \geq f(g)\} \subset (G \times \mathbb{N}_0, +)$$

is a finitely primary monoid of rank one and exponent $\max f(G)$.

2. We consider a group G with basis $E = \{e_m, e'_m \mid m \in \mathbb{N}\}$, where $\text{ord}(e_m) = \text{ord}(e'_m) = m$ for all $m \in \mathbb{N}$, whence

$$G = \bigoplus_{m \in \mathbb{N}} (\langle e_m \rangle \oplus \langle e'_m \rangle) = \bigoplus_{m \in \mathbb{N}} (\mathbb{Z}/m\mathbb{Z})^2.$$

Let $f : G \rightarrow \mathbb{N}_0$ be defined by $f(0) = 0$, $f(e) = 1$ for all $e \in E$, and $f(g) = 2$ for all $g \in G \setminus (E \cup \{0\})$. Then f satisfies all properties required in part 1, and we study $H(G, f) = H$.

3. Let $n \in \mathbb{N}$ and $a_n = (0, n) \in H$. Then $z_n = (e_n, 1) + \dots + (e_n, 1) \in Z_n(a_n)$, $z'_n = (e'_n, 1) + \dots + (e'_n, 1) \in Z_n(a_n)$, and we assert that, for every $z \in Z_n(a_n) \setminus \{z_n\}$, we have $d(z_n, z) = n$. Then we find

$$c_{\text{mon}}(H) \geq c_{\text{eq}}(H) \geq c_{\text{eq}}(a_n) \geq n,$$

whence $c_{\text{mon}}(H) = c_{\text{eq}}(H) = \infty$. Let $z \in Z_n(a_n) \setminus \{z_n\}$. Then z has the form $z = (g_1, 1) + \dots + (g_n, 1)$ with $g_1, \dots, g_n \in G$. Since $1 \geq f(g_i)$ for every $i \in [1, n]$, it follows that $\{g_1, \dots, g_n\} \subset E \cup \{0\}$. If $e_n \in \{g_1, \dots, g_n\}$, then $g_1 = \dots = g_n = e_n$, because E is a basis. Since $z \neq z_n$, we infer that $e_n \notin \{g_1, \dots, g_n\}$, whence $d(z_n, z) = n$.

4. Let $(g, n) \in H(G, f)$. First we prove that either $\max L((g, n)) = n-1$ or $d(Z_{n-1}((g, n)), Z_n((g, n))) = 3$. If $\max L((g, n)) = n-1$, then the assertion is trivial. Thus assume $\max L((g, n)) = n$. Then there is $z \in Z_n((g, n))$ of the form $z = (e_{n_1}, 1) + \dots + (e_{n_n}, 1)$ with $e_{n_1}, \dots, e_{n_n} \in E$ and $e_{n_1} + \dots + e_{n_n} = g$. Now we find $z' = (e_{n_1} + e_{n_2} + e_2, 2) + (e_2, 1) + (e_{n_3}, 1) + \dots + (e_{n_n}, 1) \in Z_{n-1}((g, n))$ and $d(Z_{n-1}((g, n)), Z_n((g, n))) \leq d(z, z') = 3$. This proves the assertion in the second case.

If $n \leq 6$, then $\max L((g, n)) \leq 6$, and thus $c_{\text{ad}}((g, n)) \leq 6$. Let now $n \geq 7$. Then there are $n' \in [3, 6]$ and $n'' \in \mathbb{N}$ such that $n = n' + 4n''$ and $(g, n) = (g, n') + n''(0, 4)$. Since we have $z_1 = 4(e_4, 1)$, $z_2 = (e_2, 1) + (e_4, 1) + (e_2 + 3e_4, 2)$, $z_3 = 2(e_2 + 2e_4, 2) \in Z((0, 4))$, $|z_1| = 4$, $|z_2| = 3$, $|z_3| = 2$, and $d(z_1, z_2) = d(z_2, z_3) = 3$, we find that $c_{\text{ad}}((g, n)) \leq \max\{3, n'\} = n' \leq 6 < \infty$.

5. By [14, Theorem 3.1.5.2.a], each finitely primary monoid of rank one is tame. \square

Note that, for the monoid H in Example 4.2, we have $c_{\text{ad}}(H) < \infty$, and therefore the question of whether the finiteness of the tame degree implies the finiteness of the adjacent catenary degree remains open. Nevertheless, the following result from [16] might be interpreted as a strong sign that the tame degree can dominate the adjacent catenary degree.

Lemma 4.3 (cf. [16, Theorem 5.1.b]). *Let H be a tame monoid. Then there exists a constant $M \in \mathbb{N}_0$ such that, for all $a \in H$ and for each two adjacent lengths $k, l \in L(a) \cap [\min L(a) + M, \max L(a) - M]$, we have $d(Z_k(a), Z_l(a)) \leq M$.*

Lemma 4.4. *Let H be an atomic monoid.*

1. *If H is half-factorial, then $c_{\text{ad}}(H) = 0$ and $c_{\text{mon}}(H) = c_{\text{eq}}(H) = c(H)$.*
2. *If $a \in H$ satisfies $|L(a)| \leq 2$, then $\mu_{\text{ad}}(a) \leq t(H)$.*

Proof. Since the arithmetic of H is determined solely by H_{red} , we may assume that H is reduced.

1. Since, for all $a \in H$ with $|L(a)| = 1$, we have no adjacent lengths, it follows that $c_{\text{ad}}(H) = 0$, and thus $c_{\text{mon}}(H) = c_{\text{eq}}(H)$. As—in this special situation—every chain of factorizations is an equal-length chain of factorizations, we get $c_{\text{eq}}(H) = c(H)$.
2. Choose $a \in H$ such that $|L(a)| \leq 2$. If $|L(a)| = 1$, then $\mu_{\text{ad}}(a) = 0$. Now suppose $|L(a)| = 2$. If $\mu_{\text{ad}}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\text{ad}}(a) > 0$. Then $\mu_{\text{ad}}(a) = \max L(a)$, and thus $\gcd(x, y) = 1$ for all $x, y \in Z(a)$ with $|x| = \min L(a)$ and $|y| = \max L(a)$. Let $x, y \in Z(a)$ with $|x| = \min L(a)$ and $|y| = \max L(a)$ and choose $u \in \mathcal{A}(H)$ such that $x \in Z(a) \cap uH^\times Z(H)$. Then there is no $y' \in Z(a) \cap uH^\times Z(H)$ with $|y'| = |y|$. Now we find

$$t(H) \geq t(a, uH^\times) \geq d(y, Z(a) \cap uH^\times Z(H)) = |y| = \max L(a) = \mu_{\text{ad}}(a). \quad \square$$

Next we formulate another variant of the catenary degree, which is somewhat similar to the adjacent catenary degree and equals it in a special situation. The main difference is that we can prove that the m -adjacent catenary degree is finite for tame monoids when m is sufficiently large.

Definition 4.5. Let H be an atomic monoid, let $a \in H$ and let $m \in \mathbb{N}$.

1. We set

$$\mu_{\text{ad},m}(a) = \sup\{k \in L(a) \mid d(Z_k(a), Z_{[k-m,k]}(a)) = k\} \quad \text{and} \quad \mu_{\text{ad},m}(H) = \sup\{\mu_{\text{ad},m}(a) \mid a \in H\}.$$

2. We define

$$c_{\text{ad},m}(a) = \sup\{d(Z_k(a), Z_{[k-m,k]}(a)) \mid k \in L(a)\}$$

as the m -adjacent catenary degree of a .

Also, $c_{\text{ad},m}(H) = \sup\{c_{\text{ad},m}(a) \mid a \in H\}$ is called the m -adjacent catenary degree of H .

Obviously, we find

$$c_{\text{ad},m}(H) \begin{cases} = 0 & m < \min \Delta(H) \\ \leq c_{\text{ad}}(H) & \\ = c_{\text{ad}}(H) & \Delta(H) = \{n\} \text{ and } n \leq m < 2n. \end{cases}$$

Since the definitions of the m -adjacent catenary degree and $\mu_{\text{ad},m}(H)$ are similar to those of the adjacent catenary degree and $\mu_{\text{ad}}(H)$, we can now prove the analog of [Theorem 3.2.3](#) for the two newly defined invariants.

Theorem 4.6. *Let H be an atomic monoid and let $m \in \mathbb{N}$. Then*

1. $\mathbf{c}_{\text{ad},m}(a) \geq \mu_{\text{ad},m}(a)$ for all $a \in H$, and $\mathbf{c}_{\text{ad},m}(H) = \mu_{\text{ad},m}(H)$.
2. $\mathbf{c}_{\text{ad},m}(H) \leq \mathbf{t}(H)$ for all $m \geq \mathbf{t}(H)$.

Proof.

1. For $m < \min \Delta(H)$, we have $\mathbf{c}_{\text{ad},m}(H) = 0 = \mu_{\text{ad},m}(H)$ by definition. Now let $m \in \mathbb{N}$ and $m \geq \min \Delta(H)$.

First we let $a \in H$ and show that $\mathbf{c}_{\text{ad},m}(a) \geq \mu_{\text{ad},m}(a)$, after which $\mathbf{c}_{\text{ad},m}(H) \geq \mu_{\text{ad},m}(H)$ follows by passing to the supremum on both sides. If $\mu_{\text{ad},m}(a) = 0$ or $\mu_{\text{ad},m}(a) = \infty$, this is trivial. Now let $\mu_{\text{ad},m}(a) = k \in \mathbb{N}$ and $[k-m, k] \cap \mathbf{L}(a) = \{l_1, \dots, l_n\}$. Then, by [Definition 4.5.2](#), $\mathbf{c}_{\text{ad},m}(a) \geq \mathbf{d}(Z_k(a), Z_{[k-m,k]}(a)) = k = \mu_{\text{ad},m}(a)$.

Now we prove $\mu_{\text{ad},m}(H) \geq \mathbf{c}_{\text{ad},m}(H)$. We must prove that $\mathbf{c}_{\text{ad},m}(a) \leq \mu_{\text{ad},m}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathbf{c}_{\text{ad},m}(a) > \mu_{\text{ad},m}(H)$. Let $k \in \mathbb{N}$ be minimal such that there is $a \in H$ with $\mathbf{c}_{\text{ad},m}(a) = \mathbf{d}(Z_k(a), Z_{[k-m,k]}(a))$. If $\mathbf{d}(Z_k(a), Z_{[k-m,k]}(a)) < k$, then there are some $x \in Z_k(a)$ and $y \in Z_{[k-m,k]}(a)$ such that $g = \gcd(x, y) \neq 1$. If $b = \pi_H(g^{-1}x)$, then $k - |g|$, $|y| - |g| \in \mathbf{L}(b) \cap [k - |g| - m, k - |g|]$ and

$$\mathbf{c}_{\text{ad},m}(a) = \mathbf{d}(Z_k(a), Z_{[k-m,k]}(a)) \leq \mathbf{d}(Z_{k-|g|}(b), Z_{[k-|g|-m, k-|g|]}(b)) \leq \mathbf{c}_{\text{ad},m}(b),$$

and, by the minimal choice of k , we infer that $\mathbf{c}_{\text{ad},m}(b) \leq \mu_{\text{ad},m}(H)$, a contradiction.

2. If H is not tame, then there is no $m \in \mathbb{N}$ with $m \geq \mathbf{t}(H)$. Thus we may assume that $\mathbf{t}(H) < \infty$. Let $m \geq \mathbf{t}(H)$. By part 1, it suffices to show that $\mu_{\text{ad},m}(a) \leq \mathbf{t}(H)$ for all $a \in H$. Let $a \in H$. If $\mu_{\text{ad},m}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\text{ad},m}(a) = k > 0$. Then we have $\mathbf{L}(a) \cap [k-m, k] = \{l_1, \dots, l_n\}$ and $\mathbf{d}(Z_k(a), Z_{l_i}(a)) = k$ for all $i \in [1, n]$. Then $\gcd(x, y) = 1$ for all $x \in Z_k(a)$ and $y \in Z_{l_i}(a)$. Now let $x \in Z_k(a)$, $y \in Z_{l_1}(a)$, and choose $u \in \mathcal{A}(H)$ such that $y \in Z(a) \cap uH^\times Z(H)$. We find

$$(4.1) \quad \begin{aligned} \mathbf{t}(H) &\geq \mathbf{t}(a, uH^\times) \geq \mathbf{d}(x, Z(a) \cap uH^\times Z(H)) \\ &= \min\{\mathbf{d}(x, Z_l(a) \cap uH^\times Z(H)) \mid l \in \mathbf{L}(a), l \neq k\} \geq \min\{k, m+1\} = k = \mu_{\text{ad},m}(a), \end{aligned}$$

since $m+1 > \mathbf{t}(H)$. \square

Another interesting observation arising from the proof of [Theorem 4.6.2](#) is the fact that the crucial inequality (4.1) might fail for $m < \mathbf{t}(H)$ for some $a \in H$ (of course, with $\mu_{\text{ad},m}(a) > 0$). Additionally, [Theorem 4.6.2](#) can never be used to bound $\mathbf{c}_{\text{ad}}(H)$, since $\mathbf{c}_{\text{ad}}(H) = \mathbf{c}_{\text{ad},m}(H)$ for $m = \min \Delta(H)$ if $|\Delta(H)| = 1$, but then $\mathbf{t}(H) \geq m+2 > m$, and therefore [Theorem 4.6.2](#) does not hold for $\mathbf{c}_{\text{ad}}(H)$.

5. APPLICATIONS TO SEMIGROUP RINGS AND GENERALIZED POWER SERIES RINGS

The arithmetic of such domains has attracted a lot of interest; for an overview, we refer to [\[1\]](#) and [\[2\]](#); and for some recent results, we refer to [\[20\]](#) and [\[21\]](#). Nevertheless, there are nearly no precise results on their arithmetic. In order to apply our monoid theoretic tools from [Section 3](#) and [\[24\]](#) to the explicit computation of various arithmetical invariants of semigroup rings and generalized power series rings, we follow a 2-step strategy. In the first step, we apply transfer principles as described in much detail in [\[14, Section 3.2\]](#), and in the second step, we make use of the monoid theoretic tools.

Definition 5.1. A monoid homomorphism $\theta : H \rightarrow B$ is called a *transfer homomorphism* if it has the following properties:

T1 $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

T2 If $a \in H$, $r, s \in B$ and $\theta(a) = rs$, then there exist $b, c \in H$ such that $\theta(b) \sim r$, $\theta(c) \sim s$, and $a = bc$.

Definition 5.2. Let $\theta : H \rightarrow B$ be a transfer homomorphism of atomic monoids and $\bar{\theta} : Z(H) \rightarrow Z(B)$ the unique homomorphism satisfying $\bar{\theta}(uH^\times) = \theta(u)B^\times$ for all $u \in \mathcal{A}(H)$. We call $\bar{\theta}$ the extension of θ to the factorization monoids.

For $a \in H$, the *catenary degree in the fibers* $\mathbf{c}(a, \theta)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any two factorizations $z, z' \in Z(a)$ with $\bar{\theta}(z) = \bar{\theta}(z')$, there exists a finite sequence of factorizations (z_0, z_1, \dots, z_k) in $Z(a)$ such that $z_0 = z$, $z_k = z'$, $\bar{\theta}(z_i) = \bar{\theta}(z)$, and $d(z_{i-1}, z_i) \leq N$ for all $i \in [1, k]$; that is, z and z' can be concatenated by an N -chain in the fiber $Z(a) \cap \bar{\theta}^{-1}(\bar{\theta}(z))$.

Also, $c(H, \theta) = \sup\{c(a, \theta) \mid a \in H\}$ is called the *catenary degree in the fibers of H* .

We briefly fix the notation concerning sequences over finite abelian groups. Let G be an additively written, finite abelian group. For a subset $A \subset G$ and an element $g \in G$, we set $-A = \{-a \mid a \in A\}$ and $A - g = \{a - g \mid a \in A\}$. Let $\mathcal{F}(G)$ be the free abelian monoid with basis G . The elements of $\mathcal{F}(G)$ are called *sequences* over G . If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \dots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \dots, g_l \in G$. For a sequence $S = g_1 \cdot \dots \cdot g_l$, we call

$$\begin{aligned} |S| &= l \text{ the length of } S, \\ \sigma(S) &= \sum_{i=1}^l g_i \in G \text{ the sum of } S, \\ \text{supp}(S) &= \{g_1, \dots, g_l\} \subset G \text{ the support of } S, \\ \Sigma(S) &= \{\sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, l]\} \subset G \text{ the set of subsums of } S, \text{ and} \\ -\Sigma(S) &= \{\sum_{i \in I} (-g_i) \mid \emptyset \neq I \subset [1, l]\} = \{-g \mid g \in \Sigma(S)\} \subset G \text{ the set of negative subsums of } S. \end{aligned}$$

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no non-trivial zero-sum subsequence, i.e. $0 \notin \Sigma(S)$, and
- a *minimal zero-sum sequence* if S is nontrivial, $\sigma(S) = 0$, and every subsequence $S' \mid S$ with $1 \leq |S'| < |S|$ is zero-sum free.

For a subset $G_0 \subset G$, we set

$$\begin{aligned} \mathcal{B}(G_0) &= \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \text{ for the block monoid over } G_0 \text{ and} \\ \mathcal{A}(G_0) &= \{S \in \mathcal{F}(G_0) \mid S \text{ minimal zero-sum sequence}\} \subset \mathcal{B}(G_0). \end{aligned}$$

Then, in fact, $\mathcal{B}(G_0)$ is an atomic monoid and $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is its set of atoms.

The *Davenport constant* $D(G_0) \in \mathbb{N}$ is defined to be the supremum of all lengths of sequences in $\mathcal{A}(G_0)$.

Definition 5.3. Let G be an additive abelian group, $G_0 \subset G$ a subset, T a monoid, $\iota : T \rightarrow G$ a homomorphism, and $\sigma : \mathcal{F}(G_0) \rightarrow G$ the unique homomorphism such that $\sigma(g) = g$ for all $g \in G_0$. Then we call

$$\mathcal{B}(G_0, T, \iota) = \{St \in \mathcal{F}(G_0) \times T \mid \sigma(S) + \iota(t) = 0\}$$

the *T -block monoid* over G_0 defined by ι .

If $T = \{1\}$, then $\mathcal{B}(G_0, T, \iota) = \mathcal{B}(G_0)$ is the block monoid of all zero-sum sequences over G_0 and if $G_0 = \{0\}$ then $\mathcal{B}(G_0, T, \iota) = [0] \times T$. Since $0 \in \mathcal{B}(G_0, T, \iota)$ is prime, the arithmetic of T and $\mathcal{B}(G_0, T, \iota)$ coincide in this situation.

Lemma 5.4. Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, let $G = \mathbf{q}(D/H)$ be its class group, let $\iota : T \rightarrow G$ be a homomorphism defined by $\iota(t) = [t]_{D/H}$, and suppose each class in G contains some prime element from P .

1. The map $\beta : H \rightarrow \mathcal{B}(G, T, \iota)$, given by $\beta(pt) = [p]_{D/H} + \iota(t) = [p]_{D/H} + [t]_{D/H}$, is a transfer homomorphism onto the T -block monoid over G defined by ι and $c(H, \beta) \leq 2$
2. The following inequalities hold:

$$\begin{aligned} c(\mathcal{B}(G, T, \iota)) &\leq c(H) && \leq \max\{c(\mathcal{B}(G, T, \iota)), c(H, \beta)\}, \\ c_{\text{mon}}(\mathcal{B}(G, T, \iota)) &\leq c_{\text{mon}}(H) && \leq \max\{c_{\text{mon}}(\mathcal{B}(G, T, \iota)), c(H, \beta)\}, \text{ and} \\ t(\mathcal{B}(G, T, \iota)) &\leq t(H) && \leq t(\mathcal{B}(G, T, \iota)) + D(G) + 1. \end{aligned}$$

In particular, the equality $c(H) = c(\mathcal{B}(G, T, \iota))$ holds if $c(\mathcal{B}(G, T, \iota)) \geq 2$, and the equality $c_{\text{mon}}(H) = c_{\text{mon}}(\mathcal{B}(G, T, \iota))$ holds if $c_{\text{mon}}(\mathcal{B}(G, T, \iota)) \geq 2$.

3. $\mathcal{L}(H) = \mathcal{L}(\mathcal{B}(G, T, \iota))$, $\Delta(H) = \Delta(\mathcal{B}(G, T, \iota))$, $\min \Delta(H) = \min \Delta(\mathcal{B}(G, T, \iota))$, and $\rho(H) = \rho(\mathcal{B}(G, T, \iota))$.
4. We set $\mathcal{B} = \{S \in \mathcal{B}(G, T, \iota) \mid 0 \nmid S\}$. Then \mathcal{B} and $\mathcal{B}(G, T, \iota)$ have the same arithmetical properties, and

$$\begin{aligned} c(\mathcal{B}) &\leq c(H) && \leq \max\{c(\mathcal{B}), c(H, \beta)\}, \\ c_{\text{mon}}(\mathcal{B}) &\leq c_{\text{mon}}(H) && \leq \max\{c_{\text{mon}}(\mathcal{B}), c(H, \beta)\}, \text{ and} \\ t(\mathcal{B}) &\leq t(H) && \leq t(\mathcal{B}) + D(G) + 1. \end{aligned}$$

In particular, the equality $c(H) = c(\mathcal{B})$ holds if $c(\mathcal{B}) \geq 2$, and the equality $c_{\text{mon}}(H) = c_{\text{mon}}(\mathcal{B})$ holds if $c_{\text{mon}}(\mathcal{B}) \geq 2$.

Additionally, $\mathcal{L}(H) = \mathcal{L}(\mathcal{B})$, $\Delta(H) = \Delta(\mathcal{B})$, $\min \Delta(H) = \min \Delta(\mathcal{B})$, and $\rho(H) = \rho(\mathcal{B})$.

Proof.

1. Follows by [14, Proposition 3.2.3.3 and Proposition 3.4.8.2].
2. The assertion on the catenary degree follows by [14, Theorem 3.2.5.5], the assertion on the monotone catenary degree by [14, Lemma 3.2.6], and the assertion on the tame degree by [14, Theorem 3.2.5.1].
3. Follows by [14, Proposition 3.2.3.5].
4. Since $\mathbf{0} \in \mathcal{B}(G, T, \iota)$ is a prime element, it defines a partition $\mathcal{B}(G, T, \iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G, T, \iota) \mid \mathbf{0} \nmid S\}$. Thus all studied arithmetical invariants coincide for \mathcal{B} and $\mathcal{B}(G, T, \iota)$. Now the assertions follow from part 2 and part 3. \square

From now on, we write monoids additively. Then, for a reduced monoid H , $H^\times = \{0\}$.

Definition 5.5. Let K be a field and H a reduced atomic monoid. Then we call

- $K[[H]] = K[[X^s \mid s \in \mathcal{A}(H)]]$ the *generalized power series ring*
- $K[H] = K[X^s \mid s \in \mathcal{A}(H)]$ the *semigroup ring*

defined by H over K .

A submonoid $H \subset \mathbb{N}_0$ such that $\mathbb{N}_0 \setminus H$ is finite is called a *numerical monoid*. Then we set $\mathcal{G}(H) = \mathbb{N}_0 \setminus H$ for the *set of gaps* of H . Next we give a characterization of smooth numerical monoids.

While some abstract finiteness results on these rings can be obtained in a quite general setting, only very little is known about the explicit behavior of the various invariants of non-unique factorization theory. For special generalized power series rings, we repeat the following finiteness result from [15] in our terminology.

Theorem 5.6 (cf. [15, Proposition 6.10 and Theorem 6.7]). *Let K be a field and $H \subset \mathbb{N}_0$ a numerical monoid. Then $K[[H]]$ has finite catenary degree and finite elasticity.*

In our further investigations, we will restrict ourselves generally to the most simple monoids possible, i.e., to numerical monoids, since even there the situation is very complicated once one begins to calculate everything in detail.

By [14, Theorem 3.7.1], the arithmetic of weakly Krull domains, e.g., a generalized power series ring or a semigroup ring defined over a finite field, can mostly be described by studying appropriate T -block monoids, i.e., $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$. Special cases of the generalized power series rings and the semigroup rings are studied in [14, Example 3.7.3, Special Case 3.2 and 3.3]. For special numerical monoids, i.e., $H = [e, \dots, 2e - 1]$ with $e \geq 2$, and the generalized power series ring $R = F[[H]]$ over a finite field F , there a transfer homomorphism from R onto H is constructed. In Lemma 5.8, we study for which monoids H , in general, this homomorphism is indeed a transfer homomorphism. Surprisingly, it turns out that one can characterize a special class of monoids, namely smooth monoids, this way.

Definition 5.7. Let (H, \leq) be a reduced atomic monoid with a total order. Then we call H *smooth* if, for all $h, b, c \in H$ with $h \geq b + c$, we have $h - b \in H$.

Lemma 5.8. *Let K be a field, (H, \leq) a reduced atomic monoid with total order, and set*

$$\phi : \begin{cases} K[[H]]^\bullet & \rightarrow & H \\ f = \sum_{h \in H} f_h X^h & \mapsto & \min\{h \in H \mid f_h \neq 0\}. \end{cases}$$

Then ϕ is a homomorphism, and ϕ is a transfer homomorphism if and only if H is smooth.

Proof. Obviously, ϕ is a homomorphism. First we assume that H is smooth. Now we must show the two axioms from Definition 5.1. **T1** is obvious, since ϕ is surjective. For **T2**, let $u = \sum_{h \in H} u_h X^h \in K[[H]]$ and $b, c \in H$ with $\phi(u) = b + c$. We set $v = u_{\phi(u)} X^b$ and $w = X^c + \sum_{h \in H} u_h u_{\phi(u)}^{-1} X^{h-b}$. Then $v, w \in K[[H]]$, $vw = u$, and $\phi(w) = c$.

Second, we assume that H is not smooth and show that axiom **T2** of Definition 5.1 fails. Since H is not smooth, there are $h, b, c \in H$ with $h > b + c$, $h - b, h - c \notin H$ and all three elements are minimal with this property. Then set $u = X^{b+c} + X^h$. Now $\phi(u) = b + c$. Let $h' \in \{b, c\}$ and set $v = X^{h'}$. Then $v \nmid u$ in $K[[H]]$, and thus **T2** does not hold. \square

Proposition 5.9. *Let $H \subset \mathbb{N}_0$ be a numerical monoid. Then the following are equivalent.*

1. *There is $m \in \mathbb{N}$ such that $H = [m, m + 1, \dots, 2m - 1]$.*
2. *$\mathcal{G}(H)$ is an interval.*
3. *H is smooth.*

Proof. Obviously, 1 and 2 are equivalent, and the implication from 1 to 3 is also clear.

3 \Rightarrow 1. If $|\mathcal{A}(H)| = 1$, then, trivially, $H = [1] = \mathbb{N}_0$. Now let $|\mathcal{A}(H)| > 1$ and $k, n \in \mathbb{N}$ be minimal such that $[k, k+n] \cap \mathcal{A}(H) = \{k, k+n\}$. We proceed by distinguishing three cases.

Case 1. $n > k$. Then $2k < k+n$, and therefore $(k+n) - k = n \in H$. This is a contradiction to $k+n \in \mathcal{A}(H)$.

Case 2. $n = k$. Then $k+n = 2k$ and this is again a contradiction to $k+n \in \mathcal{A}(H)$.

Case 3. $n < k$. By the minimality of k and n , we have that $[k, k+n] \cap H = \{k, k+n\}$. Let $m \in \mathbb{N}$ be minimal such that $mn \geq k$. If $mn > k$, then $mn \in (k, kn)$, a contradiction; therefore $mn = k$, i.e., $n \mid k$. Let $x \in H \setminus [mn, (m+1)n, \dots, (2m-1)n] \cap [0, 2mn)$. Additionally, $x > (m+a)n$ for some $a \in \mathbb{N}$, since $x \notin (mn, (m+1)n)$. Thus we find $x \in ((m+1)n, 2mn) \setminus n\mathbb{N}$. Then there exist $j \in [m+1, 2m-1]$ and $l \in [1, n-1]$ such that $x = jn + l$. Let $j' \in [m-1, 2m-1]$ be such that $j + j' = im$ for $i \geq 2$. Then we find $imn < imn + l = x + j'n \in H$. Thus we find $x + j'm - mn = (i-1)mn - l \in H$. This can now be repeated until we reach $mn + l \in H$ —a contradiction. Thus $H = [mn, (m+1)n, \dots, (2m+1)n]$ and, since H is a numerical monoid, we have $\gcd(\mathcal{A}(H)) = 1$, and therefore $n = 1$. Now the assertion follows. \square

For the study of semigroup rings, the situation is even more difficult. Since there is then no transfer homomorphism $R = F[H] \rightarrow H$; see [14, Example 3.7.3, Special Case 3.3]. Thus—even after applying the transfer principles in order to be in an easier situation—it is necessary to compute all the invariants of non-unique factorization for more general T -block monoids, $\mathcal{B}(G, T, \iota)$, where neither T nor G are trivial. In the upcoming subsections, we exploit the results from [6], [5], [24, Proposition 16] (repeated as Lemma 2.7), [24, Theorem 19.2], and Section 3 (mainly Theorem 3.2) together with recent programming techniques (see [17] and [23, Section 8]) and parallelization to explicitly compute various arithmetical invariants, namely, the elasticity, the catenary degree, the monotone catenary degree, and a bound for the tame degree of the T -block monoids associated with the studied domains, and therefore for the domains themselves.

5.1. Preliminaries about zero-sum sequences and T -block monoids.

In order to be able to describe the set of atoms of a T -block monoid precisely, we use the terminology of sequences over finite abelian groups.

For our algorithmic considerations in the forthcoming sections, it will be very useful to have some sort of order defined on the elements of a finite abelian group G . By the structure theorem for finitely generated abelian groups, there are uniquely determined $r \in \mathbb{N}_0$ and $n_1, \dots, n_r \in \mathbb{N}$ such that there is a group isomorphism $\varphi : G \rightarrow \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z}$ and $1 < n_1 \mid \dots \mid n_r$. For $i \in [1, r]$, we choose $[0, n_i)$ as a system of representatives for $\mathbb{Z}/n_i\mathbb{Z}$. Now we can compare two elements $g_1, g_2 \in G$ by comparing $\varphi(g_1)$ and $\varphi(g_2)$ with respect to the lexicographic order. For short, we simply write $g_1 \leq g_2$ respectively $g_1 \geq g_2$.

In particular, in Subsection 5.4, we will need some kind of coordinate representation for the elements of a T -block monoid, i.e., a monoid isomorphism mapping a T -block monoid onto a submonoid of $\mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z}$ for some $m, r \in \mathbb{N}_0$ and $n_1, \dots, n_r \in \mathbb{N}$. Let G be a finite abelian group, T a finitely generated monoid, and $\iota : T \rightarrow G$ a homomorphism. Let $T = D_1 \times \dots \times D_r$ be a product of finitely primary monoids $D_i \subset [p_1^{(i)}, \dots, p_{r_i}^{(i)}] \times \widehat{D_i}^\times = \widehat{D_i}$ where $r_i \in \mathbb{N}$, and the $\widehat{D_i}^\times$ are finitely generated abelian groups for $i \in [1, r]$. Then there are uniquely determined $l_i, k_i \in \mathbb{N}_0$ such that there is an isomorphism $\phi_i : \widehat{D_i}^\times \rightarrow \mathbb{Z}^{l_i} \times \mathbb{Z}/n_1^{(i)}\mathbb{Z} \times \dots \times \mathbb{Z}/n_{k_i}^{(i)}\mathbb{Z}$ with $1 < n_1^{(i)} \mid \dots \mid n_{k_i}^{(i)}$ for $i \in [1, r]$. This isomorphism can be extended to an isomorphism $\bar{\phi}_i : \widehat{D_i} \rightarrow N_0^{r_i} \times \phi_i(\widehat{D_i}^\times)$ for $i \in [1, r]$. Now there is an isomorphism $\phi = \bar{\phi}_1 \times \dots \times \bar{\phi}_r : \widehat{T} \rightarrow \bar{\phi}_1(\widehat{D_1}) \times \dots \times \bar{\phi}_r(\widehat{D_r})$. This again can be extended to an isomorphism $\bar{\varphi} : \mathcal{F}(G) \times \widehat{T} \rightarrow N_0^{|G|} \times \phi(\widehat{T})$. Now we can define the desired isomorphism by restriction of $\bar{\varphi}$ to the T -block monoid $\mathcal{B}(G, T, \iota)$ as follows:

$$(5.1) \quad \varphi = \bar{\varphi}|_{\mathcal{B}(G, T, \iota)} : \mathcal{B}(G, T, \iota) \rightarrow \bar{\varphi}(\mathcal{B}(G, T, \iota)) \subset \mathbb{N}_0^{|G|} \times \prod_{i=1}^r \left(\mathbb{N}_0^{r_i} \times \mathbb{Z}^{l_i} \times \prod_{j=1}^{k_i} \mathbb{Z}/n_j^{(i)}\mathbb{Z} \right).$$

5.2. The set of atoms $\mathcal{A}(G)$ of a block monoid.

Based on ideas from [17], we give an algorithm for the computation of the set of atoms $\mathcal{A}(G)$ for a finite additive abelian group G . The problem of computing $\mathcal{A}(G)$ grows exponentially in terms of $|G|$, but, for very small groups as the ones involved in Subsection 5.5, it can be easily performed—sometimes even by hand. Unfortunately, we have to do some sort of brute force search in the set of all $S \in \mathcal{F}(G)$ with $|S| \leq D(G)$. But with the algorithm presented below, we can avoid most of the redundant checks and therefore speed up the computation dramatically.

Since modular arithmetic on vectors with multiple coordinates is quite inefficient, it is necessary for a fast execution of the RAS, Algorithm 1, to pre-compute the sums $g + g'$. This can be done once in the

Algorithm 1 Recursive Atom Search: $A \leftarrow \text{RAS}(A, S, \Sigma, B)$

```

for all  $g \in B$  do
   $S' \leftarrow Sg$ 
  if  $g \leq -\sigma(S')$  then
     $A \leftarrow A \cup \{S'(-\sigma(S'))\}$ 
  end if
   $\Sigma' \leftarrow \Sigma$ 
   $B' \leftarrow \emptyset$ 
  for all  $g' \in B$  do
    if  $g + g' \in \Sigma$  then
       $\Sigma' \leftarrow \Sigma' \cup \{g'\}$ 
    else
       $B' \leftarrow B' \cup \{g'\}$ 
    end if
  end for
  if  $|B'| > 0$  then
     $A \leftarrow \text{RAS}(A, S', \Sigma', B')$ 
  end if
end for
return  $A$ 

```

Algorithm 2 Atoms Computation Algorithm 1: $\mathcal{A}(G) \leftarrow \text{ACA1}(G)$

```

 $A \leftarrow \{0\}$ 
for all  $g \in G \setminus \{0\}$  do
  if  $g \leq -g$  then
     $A \leftarrow A \cup \{g(-g)\}$ 
  end if
   $\Sigma \leftarrow \{0, g\}$ 
   $B \leftarrow G \setminus \{0, -g\}$ 
   $S \leftarrow g$ 
  if  $|B| > 0$  then
     $A \leftarrow \text{RAS}(A, S, \Sigma, B)$ 
  end if
end for
return  $A$ 

```

ACA1, [Algorithm 2](#), before the main loop. For additional details on speeding up these types of algorithms by special alignment of the pre-computed data and on the parallelization aspects, the reader is referred to [\[17, Section 3\]](#).

5.3. The set of atoms of a T -block monoid.

Lemma 5.10. *Let G be a finite additive abelian group, T a reduced atomic monoid, $\iota : T \rightarrow G$ a homomorphism, and $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$ the T -block monoid over G defined by ι . Furthermore, suppose each class in G contains some $p \in P$, and let $\bar{\iota} : \mathbb{Z}(T) \rightarrow \mathcal{F}(G)$ be the homomorphism generated by the extension of ι onto $\mathbb{Z}(T)$ such that, for a factorization $z = a_1 \cdots a_n \in \mathbb{Z}(T)$ with $a_i \in \mathcal{A}(T)$ for $i \in [1, n]$, we have $\bar{\iota}(z) = \iota(a_1) \cdots \iota(a_n)$.*

Then we have

$$(5.2) \quad \mathcal{A}(\mathcal{B}(G, T, \iota)) = \{S\pi(z) \mid S \in \mathcal{F}(G), z \in \mathbb{Z}(T), S\bar{\iota}(z) \in \mathcal{A}(G), \#n \geq 2 : \exists S_i \in \mathcal{F}(G), z_i \in \mathbb{Z}(T) \text{ with } S_i\bar{\iota}(z_i) \in \mathcal{A}(G) \text{ for } i \in [1, n] : S_1\pi(z_1) \cdots S_n\pi(z_n) = S\pi(z)\}$$

Proof. Clearly, every atom $a \in \mathcal{A}(\mathcal{B}(G, T, \iota))$ is of the form $a = S\pi(z)$ with $S \in \mathcal{F}(G)$, $z \in \mathbb{Z}(T)$, and $S\bar{\iota}(z) \in \mathcal{A}(G)$. Now suppose we have $n \in [2, D(G)]$, $S_i \in \mathcal{F}(G)$, $z_i \in \mathbb{Z}(T)$, $S_i\bar{\iota}(z_i) \in \mathcal{A}(G)$ for $i \in [1, n]$ and $S\pi(z) = S_1\pi(z_1) \cdots S_n\pi(z_n)$. Obviously then, $a \notin \mathcal{A}(\mathcal{B}(G, T, \iota))$. Now the other inclusion is obvious. \square

In general, it is very hard to calculate $\mathcal{A}(\mathcal{B}(G, T, \iota))$ explicitly by the characterization in [\(5.2\)](#). But if we restrict ourselves to a finite group G and a finitely generated reduced monoid T such that $\mathcal{A}(G)$,

$\mathcal{A}(T)$, and $\iota(a)$ for $a \in \mathcal{A}(T)$ are all known explicitly, we can formulate the ACA2, [Algorithm 3](#), for the computation of the set of atoms of a T -block monoid.

Algorithm 3 Atoms Computation Algorithm 2: $\mathcal{A}(\mathcal{B}(G, T, \iota)) \leftarrow \text{ACA2}(G, T, \mathcal{A}(G), \mathcal{A}(T), \iota)$

```

 $A \leftarrow \emptyset$ 
 $D \leftarrow 0$ 
for all  $S \in \mathcal{A}(G)$  do
  if  $|S| > D$  then
     $D \leftarrow |S|$ 
  end if
   $A \leftarrow A \cup \{(S, 1)\}$ 
end for
 $F_0 \leftarrow \emptyset$ 
for all  $a \in \mathcal{A}(T)$  do
  for all  $(S, 1) \in A$  do
    if  $\iota(a) \mid S$  then
       $F_0 \leftarrow F_0 \cup \{(\iota(a)^{-1}S, a)\}$ 
    end if
  end for
end for
 $E \leftarrow \emptyset$ 
 $n \leftarrow 1$ 
while  $n < D$  and  $F_{n-1} \neq \emptyset$  do
   $E \leftarrow E \cup F_{n-1}$ 
   $E \leftarrow EF_0$ 
   $F_n \leftarrow \emptyset$ 
  for all  $a \in \mathcal{A}(T)$  do
    for all  $(S, b) \in A$  do
      if  $\iota(a) \mid S$  then
         $F_n \leftarrow F_n \cup \{(\iota(a)^{-1}S, ab)\}$ 
      end if
    end for
  end for
   $n \leftarrow n + 1$ 
end while
return  $A \cup F_0 \cup \dots \cup F_{n-1}$ 

```

5.4. Computing arithmetical invariants of a T -block monoid.

Throughout this section, we implicitly use the isomorphism defined in (5.1). Thus we only have to work with submonoids

$$S \subset \mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \dots \times \mathbb{Z}/n_r\mathbb{Z} \quad \text{with } m, r \in \mathbb{N}_0 \text{ and } n_1, \dots, n_r \in \mathbb{N}$$

such that $S \cong \varphi(\mathcal{B}(G, T, \iota))$, where G is an additively written finite abelian group, T is a product of finitely many reduced finitely primary monoids of rank 1, $\iota : G \rightarrow T$ is a homomorphism, and φ is the isomorphism defined in (5.1). If T is not the product of finitely many reduced finitely primary monoids of rank 1, then T would not be finitely generated. Now we know $\mathcal{A}(S)$ explicitly, since, obviously, $\mathcal{A}(S) = \varphi(\mathcal{A}(\mathcal{B}(G, T, \iota)))$ and $\mathcal{A}(\mathcal{B}(G, T, \iota))$ can be computed explicitly by the ACA2; see [Algorithm 3](#).

For the computation of the tame degree, we use the definition of the distance of factorizations and [24, Theorem 19.2]; for additional reference on this computation, see [6, Section 4].

Now we are ready to describe the computation step by step.

5.4.1. Finding the elements of $\mathcal{A}(\sim_S)$.

The first step is finding the elements of $\mathcal{A}(\sim_S)$ explicitly. Unfortunately, this is a very hard task. Probably, the most efficient way is the following one as described in [5, Sections 1 and 2].

1. Since we know $\mathcal{A}(S)$ explicitly, we can write the atoms of S in their coordinates as vectors:

$$\mathcal{A}(S) = \{(a_1^{(1)}, \dots, a_m^{(1)}, a_{m+1}^{(1)} \bmod n_1, \dots, a_{m+r}^{(1)} \bmod n_r), \dots\}.$$

2. By [5, Section 2], finding the elements of $\mathcal{A}(\sim_S)$ is equivalent to determining the minimal positive solutions of the following system of linear diophantine equations:

$$(5.3) \quad \begin{array}{ccccccccc} x_1 a_1^{(1)} & + & \dots & + & x_k a_1^{(k)} & - & y_1 a_1^{(1)} & - & \dots & - & y_k a_1^{(k)} & = & 0 \\ \vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots \\ x_1 a_m^{(1)} & + & \dots & + & x_k a_m^{(k)} & - & y_1 a_m^{(1)} & - & \dots & - & y_k a_m^{(k)} & = & 0 \\ x_1 a_{m+1}^{(1)} & + & \dots & + & x_k a_{m+1}^{(k)} & - & y_1 a_{m+1}^{(1)} & - & \dots & - & y_k a_{m+1}^{(k)} & \equiv & 0 \pmod{n_1} \\ \vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots \\ x_1 a_{m+r}^{(1)} & + & \dots & + & x_k a_{m+r}^{(k)} & - & y_1 a_{m+r}^{(1)} & - & \dots & - & y_k a_{m+r}^{(k)} & \equiv & 0 \pmod{n_r} \end{array}$$

We write a solution $(x_1, \dots, x_k, y_1, \dots, y_k)$ as $((x_1, \dots, x_k), (y_1, \dots, y_k))$.

3. Again, by [5, Section 2] and [27, Section 2], finding the set of minimal positive solutions is equivalent to finding the set of minimal positive solutions for the following enlarged system and then projecting back by the map and removing the zero element (if appearing after the projection) from the set of solutions:

$$(5.4) \quad \begin{array}{ccccccccccc} x_1 a_1^{(1)} & + & \dots & - & y_1 a_1^{(1)} & - & \dots & & & & = & 0 \\ \vdots & & & & \vdots & & & & & & \vdots & \\ x_1 a_m^{(1)} & + & \dots & - & y_1 a_m^{(1)} & - & \dots & & & & = & 0 \\ x_1 a_{m+1}^{(1)} & + & \dots & - & y_1 a_{m+1}^{(1)} & - & \dots & + & x_{k+1} n_1 & - & y_{k+1} n_1 & = & 0 \\ \vdots & & & & \vdots & & & & \vdots & & \vdots & & \vdots \\ x_1 a_{m+r}^{(1)} & + & \dots & - & y_1 a_{m+r}^{(1)} & - & \dots & + & x_{k+r} n_r & - & y_{k+r} n_r & = & 0 \end{array}$$

$$\Phi : \left\{ \begin{array}{ccc} \mathbb{N}_0^{k+r} \times \mathbb{N}_0^{k+r} & \rightarrow & \mathbb{N}_0^k \times \mathbb{N}_0^k \\ ((x_1, \dots, x_{k+r}), (y_1, \dots, y_{k+r})) & \mapsto & ((x_1, \dots, x_k), (y_1, \dots, y_k)) \end{array} \right\}.$$

One of the most efficient algorithms for finding these solutions is due to Contejean and Devie; see [7]. Nevertheless, this might take a very long time since the problem of determining the set of all minimal non-negative solutions of a system of linear diophantine equations is well known to be NP-complete.

5.4.2. Removing unnecessary elements.

Clearly, elements of the form $((0, \dots, 0, 1, 0, \dots, 0), (0, \dots, 0, 1, 0, \dots, 0))$ are minimal solutions. But as elements of $\mathcal{A}(\sim_S)$, these elements do not carry any information about the arithmetic of S . Therefore we may simply drop them. Since, for any two factorizations, $(x, y) \in Z(S)$ is equivalent to $(y, x) \in Z(S)$, we may also reduce the number of pairs by a factor of two. This smaller set will be denoted by $\mathcal{A}(\sim_S)^* = \{((x_1, \dots, x_k), (y_1, \dots, y_k)), \dots\}$.

5.4.3. Computing the elasticity.

By our finiteness assumptions on T , i.e., since T is finitely generated, we know this set is finite. Thus we can simply compute the elasticity using [24, Proposition 14.2] as follows:

$$\rho(S) = \max \left\{ \frac{x_1 + \dots + x_k}{y_1 + \dots + y_k}, \frac{y_1 + \dots + y_k}{x_1 + \dots + x_k} \mid ((x_1, \dots, x_k), (y_1, \dots, y_k)) \in \mathcal{A}(\sim_S)^* \right\}.$$

5.4.4. Computing the catenary degree.

By Lemma 2.7.2, we need only consider elements $a \in S$ such that their factorizations appear as part of an element of $\mathcal{A}(\sim_S)$ and such that their sets of factorizations consist of more than one \mathcal{R} -equivalence class. Then we get the catenary degree by taking the maximum over $\mu(a)$ for all those $a \in S$.

5.4.5. Computing the tame degree.

After having computed $Z(a)$ for all $a \in S$ such that $\mathcal{A}_a(\sim_S) \neq \emptyset$, we can apply [24, Theorem 19.1] for every $u \in \mathcal{A}(S)$. Since there are only finitely many, we get the tame degree as the maximum of these values.

Algorithm 4 Recursive \mathcal{R} -Class Finder: $\mathcal{R} \leftarrow \text{RCF}(\mathcal{R}, Z = \{z_1, \dots, z_n\})$

```

 $r \leftarrow \{z_1\}$ 
 $Z \leftarrow Z \setminus \{z_1\}$ 
 $n \leftarrow n - 1$ 
 $Z = \{z_1, \dots, z_n\}$  {renumber}
 $i \leftarrow 1$ 
while  $i < n$  do
  for  $i = 1$  to  $n$  do
    if  $\gcd(z_i, x) \neq 1$  for some  $x \in r$  then
       $r \leftarrow r \cup \{z_i\}$ 
       $Z \leftarrow Z \setminus \{z_i\}$ 
       $n \leftarrow n - 1$ 
       $Z = \{z_1, \dots, z_n\}$  {renumber}
      break
    end if
  end for
end while
 $\mathcal{R} \cup \{r\}$ 
if  $Z \neq \emptyset$  then
   $\mathcal{R} \leftarrow \text{RCF}(\mathcal{R}, Z)$ 
end if
return  $\mathcal{R}$ 

```

Algorithm 5 Catenary degree Computation Algorithm: $c(S) \leftarrow \text{CCA}(\mathcal{A}(S), \mathcal{A}(\sim_S)^*)$

```

 $A \leftarrow \emptyset$ 
for all  $(x, y) \in \mathcal{A}(\sim_S)^*$  do
   $A \leftarrow A \cup \{\pi(x)\}$ 
end for
 $c \leftarrow 0$ 
for all  $a \in A$  do
   $\mathcal{R}_a \leftarrow \text{RCF}(Z(a))$ 
  if  $|\mathcal{R}_a| > 1$  then
     $\mu \leftarrow \min\{|x| \mid \mathcal{R}_a\}$ 
    if  $c < \mu$  then
       $c \leftarrow \mu$ 
    end if
  end if
end for
return  $c$ 

```

5.4.6. Computing the monotone catenary degree.

For computing the monotone catenary degree, we compute the equal catenary degree $c_{\text{eq}}(S)$ and the adjacent catenary degree $c_{\text{ad}}(S)$. We start with the adjacent catenary degree and proceed like in 5.4.1. We use the fact that $\sim_{S, \text{mon}} = \{(x, y) \in \sim_S \mid |x| \leq |y|\}$ and again [5, Section 2]. Now finding the elements of $\mathcal{A}(\sim_{S, \text{mon}})$ is equivalent to determining the minimal positive solutions of a system of linear diophantine equations.

Before we construct this finite system of linear diophantine equations explicitly, we formulate a short lemma.

Lemma 5.11. *Let H be a finitely generated monoid.*

Then $\sim_{H, \text{mon}}$ is a finitely generated Krull monoid.

Proof. Let H be a finitely generated monoid. Since $\sim_H \subset Z(H) \times Z(H)$ is then a saturated submonoid of a finitely generated monoid, \sim_H is finitely generated by [14, Proposition 2.7.5]. Now assume \sim_H has $n \in \mathbb{N}$ generators. Then the atoms of \sim_H can be described as the minimal solutions of a system of finitely many, say k , linear diophantine equations in $2n$ variables as in step 5.4.1 above. Then the atoms of $\sim_{H, \text{mon}}$ can be described as the minimal solutions of a system of $k + 1$ linear diophantine equations in $2n + 1$ variables—see below for the explicit description of this system of linear diophantine equations. Thus H is a finitely generated Krull monoid by [14, Theorem 2.7.14]. \square

The system is (5.3), with one additional variable z and one equation, namely,

$$x_1 + \dots + x_k - y_1 - \dots - y_k + z = 0.$$

The coefficients at z are zero in all other equations. Now we have two possibilities.

- Either we proceed by the same steps as in 5.4.1 and solve this directly
- or we use the incremental version of the algorithm of Devie and Contejeud (see [7, Section 9]) and the set $\mathcal{A}(\sim_S)$, which we already computed in 5.4.1.

Next we can reduce the set of relations which we must consider, as in 5.4.2. By Theorem 3.2.4, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\text{mon}}) \neq \emptyset$. Then we get the adjacent catenary degree by taking the maximum over $\mu_{\text{ad}}(a)$ for all those a . For the computation of the equal catenary degree, we must know the elements of $\mathcal{A}(\sim_{S,\text{eq}})$. But these are already known, since $\mathcal{A}(\sim_{S,\text{eq}}) \subset \mathcal{A}(\sim_{S,\text{mon}})$. Here we can again reduce the set of relations which we must consider, as in 5.4.2. By Theorem 3.2.4, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\text{eq}}) \neq \emptyset$ and $|\mathcal{R}_{a,k}| > 1$ for some $k \in \mathbf{L}(a)$. Now this can be done by applying the RCF, Algorithm 4, to $Z_k(a)$ instead of $Z(a)$. Then we get the equal catenary degree by taking the maximum over $\mu_{\text{eq}}(a)$ for all those a .

Now we find the monotone catenary degree by $c_{\text{mon}}(S) = \max\{c_{\text{ad}}(S), c_{\text{eq}}(S)\}$.

5.4.7. Reducing the computation time for the catenary degree.

If we are only interested in the computation of the catenary degree, we can speed up the very time consuming computations in Step 5.4.1 in the following way. In favor of Lemma 2.7.2, we may restrict our search for minimal solutions of the system of linear diophantine equations (5.4) to solutions $(x_1, \dots, x_{k+r}, y_1, \dots, y_{k+r})$ such that $\sum_{i=1}^k x_i \leq c(S)$ and $\sum_{i=1}^k y_i \leq c(S)$. Of course, we do not know $c(S)$ a priori, but we may replace it with any upper bound—the better the bound, the faster the computation. In our special situation of T -block monoids, we can find a reasonably good bound by [14, Theorem 3.6.4.1] and by [14, Proposition 3.6.6]. Formulated in our terminology, these results read as follows.

Theorem 5.12. *Let G be an additively written abelian group, T a reduced finitely generated monoid, $\iota : T \rightarrow G$ a homomorphism, and $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$ the T -block monoid over G defined by ι . Then*

1. $\rho(\mathcal{B}(G, T, \iota), \mathcal{F}(G) \times T) \leq \rho(T)$.
2. $c(\mathcal{B}(G, T, \iota)) \leq \rho(T)D(G) \max\{c(T), D(G)\}$.

Now we set $C = \rho(T)D(G) \max\{c(T), D(G)\}$ for the upper bound. Though this does not speed up the search for minimal solutions itself that much, it is a very efficient (additional) termination criterion in our variant of the algorithm due to Contejean and Devie; for reference on the originally proposed algorithm, see [7].

Unfortunately, this method has one drawback for the computation of the elasticity and the tame degree. As we no longer compute all minimal solutions to our system of linear diophantine equations, we no longer compute all elements in $\mathcal{A}(\sim_S)$, and therefore we cannot compute more than a lower bound for the elasticity in Step 5.4.3 and for the tame degree in Step 5.4.5.

5.4.8. Computing the elasticity from an appropriate subset of $\mathcal{A}(\sim_S)$.

In [8], Domenjou proposed an algorithm for computing the set of minimal solutions of a system of linear diophantine equations, which computes the set of minimal solutions with minimal support in a first step. All other minimal solutions can then be found by “appropriate” linear combinations of them using non-negative rational coefficients. With this interesting fact in mind, we consider the following lemma.

Definition 5.13. Let H be an atomic monoid. For $x \in Z(H)$, we set

$$\text{supp}(x) = \{u \in \mathcal{A}(H_{\text{red}}) \mid u \mid x\}.$$

Lemma 5.14. *Let H be a finitely generated monoid. Then*

$$\rho(H) = \sup \left\{ \left\lfloor \frac{|x|}{|y|} \right\rfloor \mid (x, y) \in \mathcal{A}'(\sim_H) \right\},$$

where $\mathcal{A}'(\sim_H) = \{(x, y) \in \mathcal{A}(\sim_H) \mid \text{supp}(x) \cup \text{supp}(y) \text{ is minimal}\}$.

Proof. Let $(x, y) \in \mathcal{A}(\sim_H)$. Then there are $n \in \mathbb{N}$, $(x_i, y_i) \in \mathcal{A}'(\sim_H)$, and $q_i \in \mathbb{Q}$ with $0 \leq q_i < 1$ for $i \in [1, n]$ such that

$$(x, y) = \prod_{i=1}^n (x_i, y_i)^{q_i}.$$

Such a decomposition exists, since the equivalent one exists for the set of solutions of the associated system of linear diophantine equations, see [8, Theorem 3]. When we pass to the lengths, we find $|x| = \sum_{i=1}^n q_i |x_i|$ and $|y| = \sum_{i=1}^n q_i |y_i|$. This yields

$$\frac{|x|}{|y|} \cdot |y| = |x| = \sum_{i=1}^n q_i |x_i| = \sum_{i=1}^n q_i \frac{|x_i|}{|y_i|} |y_i| \leq \max_{i=1}^n \frac{|x_i|}{|y_i|} \sum_{i=1}^n q_i |y_i| = \max_{i=1}^n \frac{|x_i|}{|y_i|} \cdot |y|.$$

Thus we find

$$\frac{|x|}{|y|} \leq \max_{i=1}^n \frac{|x_i|}{|y_i|}.$$

Since $\mathcal{A}'(\sim_H) \subset \mathcal{A}(\sim_H)$, the assertion now follows by [24, Proposition 14.2]. \square

Thus we can restrict ourselves to minimal solutions with minimal support for computing the elasticity. As far as computational performance is concerned, the most interesting point of this approach is that there are straightforward optimizations of Domenjoud's algorithm for symmetric systems of linear diophantine equations like the one in (5.4).

5.5. Three explicit examples.

Let $p \in \mathbb{P}$ be a prime. Let $R = \mathbb{F}_p[X^{e_1}, X^{e_2}]$ with $e_1, e_2 \geq 2$. Then R is a one-dimensional noetherian domain with integral closure $\widehat{R} = \mathbb{F}_p[X]$ and conductor $\mathfrak{f} = (R : \widehat{R}) = X^f \widehat{R}$, where $X \in \widehat{R}$ is a prime element and f is the frobenius number of the numerical monoid generated by e_1 and e_2 . Thus R is an order in the Dedekind domain \widehat{R} , and $X\widehat{R}$ is the only maximal ideal of \widehat{R} containing \mathfrak{f} . Furthermore, $\widehat{R}^\times = R^\times = \mathbb{F}_p^\times$. By the computations in [14, Special case 3.2 in Example 3.7.3], we have $G = \text{Pic}(R) \cong \mathbb{F}_p$.

For a more detailed presentation of the explicit computations, the reader is referred to [25, Section 2.3.5].

5.5.1. $\mathbb{F}_3[X^2, X^3]$.

Now let $p = 3$. Since $|G| = 3$, we write $G = \{\mathbf{0}, \mathbf{e}, \mathbf{e}'\}$. Clearly—or by applying the ACA1, see Algorithm 2—we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{ee}', \mathbf{e}^3, \mathbf{e}'^3\}$. Now we apply [14, Theorem 3.7.1] and switch to the block monoid, which is a T -block monoid over G , say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 3\}, g \in G\}$ and ι is the uniquely determined homomorphism $\iota : G \rightarrow T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 3\}$ and $g \in G$.

Now we apply the ACA2, see Algorithm 3, and find

$$\begin{aligned} \mathcal{A}(\mathcal{B}(G, T, \iota)) = \{ & (\mathbf{0}, 1), (\mathbf{ee}', 1), (\mathbf{e}^3, 1), (\mathbf{e}'^3, 1), (1, X^2 \mathbf{0}), (1, X^3 \mathbf{0}), (\mathbf{e}, X^2 \mathbf{e}'), (\mathbf{e}, X^3 \mathbf{e}'), \\ & (\mathbf{e}', X^2 \mathbf{e}), (\mathbf{e}', X^3 \mathbf{e}), (\mathbf{e}^2, X^2 \mathbf{e}), (\mathbf{e}^2, X^3 \mathbf{e}), (\mathbf{e}'^2, X^2 \mathbf{e}'), (\mathbf{e}'^2, X^3 \mathbf{e}') \}. \end{aligned}$$

Using the construction from the beginning of Subsection 5.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^4 \times \mathbb{Z}/3\mathbb{Z}.$$

Then, for the set of atoms, we find

$$\begin{aligned} \mathcal{A}(S) = \{ & (1, 0, 0, 0, \bar{0}), (0, 1, 1, 0, \bar{0}), (0, 3, 0, 0, \bar{0}), (0, 0, 3, 0, \bar{0}), (0, 0, 0, 2, \bar{0}), \\ & (0, 0, 0, 3, \bar{0}), (0, 1, 0, 2, \bar{2}), (0, 1, 0, 3, \bar{2}), (0, 0, 1, 2, \bar{1}), (0, 0, 1, 3, \bar{1}), \\ & (0, 2, 0, 2, \bar{1}), (0, 2, 0, 3, \bar{1}), (0, 0, 2, 2, \bar{2}), (0, 0, 2, 3, \bar{2}) \}. \end{aligned}$$

Since the atom $(1, 0, 0, 0, \bar{0})$ is prime, we can restrict to a monoid $\bar{S} \subset \mathbb{N}_0^3 \times \mathbb{Z}/3\mathbb{Z}$. Now we can find everything by using the algorithms presented at the end of Subsection 5.4. Even in the modified version of the algorithm in Step 5.4.1—here the bound is 13.5—we find about 7,500 minimal representations to consider after the reduction in Step 5.4.2.

From those, we get $\mathbf{c}(\mathbb{F}_3[X^2, X^3]) = 3$ in Step 5.4.4. Since we did not compute all minimal solutions, we find $\mathbf{t}(\mathbb{F}_3[X^2, X^3]) \geq 4$ in Step 5.4.5.

By using the alternative approach from Step 5.4.8, we find $\rho(\mathbb{F}_3[X^2, X^3]) = \frac{5}{2}$. Note that this particular result on the elasticity can also be obtained by [14, Example 3.7.3, Special Case 3.2].

5.5.2. $\mathbb{F}_2[X^2, X^3]$.

Let $p = 2$. Then $|G| = 2$, so write $G = \{\mathbf{0}, \mathbf{e}\}$. Obviously—or by applying the ACA1, see [Algorithm 2](#)—we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}^2\}$. Now we apply [\[14, Theorem 3.7.1\]](#) as in the case $p = 3$ and switch to the block monoid, which is a T -block monoid over G , say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 3\}, g \in G\}$ and ι is the uniquely determined homomorphism $\iota : G \rightarrow T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 3\}$ and $g \in G$.

Now we apply the ACA2, see [Algorithm 3](#), as before and find

$$\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{(\mathbf{0}, 1), (\mathbf{e}^2, 1), (1, X^2 \mathbf{0}), (1, X^3 \mathbf{0}), (\mathbf{e}, X^2 \mathbf{e}), (\mathbf{e}, X^3 \mathbf{e})\}.$$

Using the construction from the beginning of [Subsection 5.4](#), we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1, 0, 0, \bar{0}), (0, 2, 0, \bar{0}), (0, 0, 2, \bar{0}), (0, 0, 3, \bar{0}), (0, 1, 2, \bar{1}), (0, 1, 3, \bar{1})\}.$$

Since the atom $(1, 0, 0, \bar{0})$ is prime, we can use the same arguments as in [Lemma 5.4.4](#) and restrict to a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with a reduced set of atoms.

Since, in this case, [Step 5.4.1](#) can be performed easily without any bound, we compute all atoms. Given this list, we immediately find $\rho(\mathbb{F}_2[X^2, X^3]) = 2$ in [Step 5.4.3](#). Note that this particular result on the elasticity can also be obtained by [\[14, Example 3.7.3, Special Case 3.2\]](#).

Now we proceed with [Step 5.4.4](#) and we deduce $c(\mathbb{F}_2[X^2, X^3]) = 3$ and $D(\mathbb{F}_2) + 1 + t(S) = 6 \geq t(\mathbb{F}_2[X^2, X^3]) \geq t(S) = 3$.

Next, we compute the monotone catenary degree. For this, we proceed as in [Step 5.4.6](#) and find $c_{\text{ad}}(S) = 3$ and $c_{\text{eq}}(S) = 3$, and thus $c_{\text{mon}}(\mathbb{F}_2[X^2, X^3]) = c_{\text{mon}}(S) = 3$.

5.5.3. $\mathbb{F}_2[X^2, X^5]$.

The results in this case differ slightly from then ones we obtained above. We have $|G| = 2$, say $G = \{\mathbf{0}, \mathbf{e}\}$. Again, we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}^2\}$. Now we apply [\[14, Theorem 3.7.1\]](#) as before and switch to the block monoid, which is a T -block monoid over G , say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 5\}, g \in G\}$ and ι is the uniquely determined homomorphism $\iota : G \rightarrow T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 5\}$ and $g \in G$.

Now we apply the ACA2, see [Algorithm 3](#), as before and find

$$\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{(\mathbf{0}, 1), (\mathbf{e}^2, 1), (1, X^2 \mathbf{0}), (1, X^5 \mathbf{0}), (\mathbf{e}, X^2 \mathbf{e}), (\mathbf{e}, X^5 \mathbf{e})\}.$$

Using the construction from the beginning of [Subsection 5.4](#), we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1, 0, 0, \bar{0}), (0, 2, 0, \bar{0}), (0, 0, 2, \bar{1}), (0, 0, 5, \bar{1}), (0, 1, 2, \bar{1}), (0, 1, 5, \bar{1})\}.$$

Since the atom $(1, 0, 0, \bar{0})$ is prime, we can use the same arguments as in [Lemma 5.4.4](#) and restrict to a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with a reduced set of atoms.

Since, in this case, [Step 5.4.1](#) can be performed without any bound, we compute all atoms. Now, we find a list of 25 atoms after [Step 5.4.2](#). Given this list, we immediately find $\rho(\mathbb{F}_2[X^2, X^5]) = 3$ in [Step 5.4.3](#). Now we proceed with [Step 5.4.4](#) and obtain $t(S) = 4$, $c(\mathbb{F}_2[X^2, X^5]) = 5$, and $D(\mathbb{F}_2) + 1 + t(S) = 7 \geq t(\mathbb{F}_2[X^2, X^5]) \geq \max\{t(S), c(\mathbb{F}_2[X^2, X^5])\} = 5$. Next, we compute the monotone catenary degree. For this, we proceed as in [Step 5.4.6](#) and start with the adjacent catenary degree. We find $c_{\text{ad}}(S) = 5$. Next we compute the equal catenary degree and find $c_{\text{eq}}(S) = 6$. Now we find $c_{\text{mon}}(\mathbb{F}_2[X^2, X^5]) = c_{\text{mon}}(S) = 6 > 5 = c(\mathbb{F}_2[X^2, X^5])$.

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