ARITHMETIC-PROGRESSION-WEIGHTED SUBSEQUENCE SUMS

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Abstract. Let G be an abelian group, let S be a sequence of terms $s_1, s_2, \ldots, s_n \in G$ not all contained in a coset of a proper subgroup of G, and let W be a sequence of n consecutive integers. Let

$$W \odot S = \{w_1 s_1 + \ldots + w_n s_n : \; w_i \text{ a term of } W, \, w_i \neq w_j \text{ for } i \neq j\},$$

which is a particular kind of weighted restricted sumset. We show that $|W \odot S| \geq \min\{|G| - 1, \, n\}$, that $W \odot S = G$ if $n \geq |G| + 1$, and also characterize all sequences S of length $|G|$ with $W \odot S \neq G$. This result then allows us to characterize when a linear equation

$$a_1 x_1 + \ldots + a_r x_r \equiv \alpha \mod n,$$

where $\alpha, a_1, \ldots, a_r \in \mathbb{Z}$ are given, has a solution $(x_1, \ldots, x_r) \in \mathbb{Z}^r$ modulo n with all x_i distinct modulo n. As a second simple corollary, we also show that there are maximal length minimal zero-sum sequences over a rank 2 finite abelian group $G \cong C_{n_1} \oplus C_{n_2}$ (where $n_1 \mid n_2$ and $n_2 \geq 3$) having k distinct terms, for any $k \in [3, \min\{n_1 + 1, \exp(G)\}]$. Indeed, apart from a few simple restrictions, any pattern of multiplicities is realizable for such a maximal length minimal zero-sum sequence.

1. Introduction

Let G be an abelian group and let S be a sequence of terms from G. It is a classical problem in additive number theory to study which elements from G can be represented as a sum of some subsequence of S (possibly of predetermined length). To make this formal, we let $\Sigma(S)$ denote the set of all elements from G that are the sum of terms from some non-empty subsequence of S, and we let $\Sigma_n(S)$, where $n \geq 0$ is an integer, denote the set of all elements from G that are the sum of terms from some n-term subsequence of S. Throughout this paper, we use the multiplicative standards from [22] [21] [17] for subsequence sum notation, with all formal definitions given in the next section and notation in the introduction kept to a minimum.

The Davenport constant $\mathsf{D}(G)$, which is the minimal length of a sequence from G that guarantees a subsequence with sum zero, i.e., that $0 \in \Sigma(S)$, is perhaps the most famous and well-studied subsequence sum question [47] [22]. Other examples include the Erdős-Ginzburg-Ziv Theorem [14] [22] [39], which states that a sequence S with length $|S| \geq 2|G| - 1$ guarantees $0 \in \Sigma_{|G|}(S)$, the now proven Kemnitz Conjecture [45] [22], which states that $0 \in \Sigma_n(S)$ for $|S| \geq 4n - 3$ when $G \cong C_n \oplus C_n$ is a rank 2 finite abelian group, and the Olson constant, which is analogous to the Davenport Constant only for sets instead of sequences [8] [18] [41]. Related to the Olson Constant is the Critical Number, which is the minimal cardinality of a subset A of G needed to guarantee that *every* element of G can be represented as a sum of distinct elements from A [15], i.e., that $\Sigma(A) = G$. See [27] [12] [40] for a handful of more recent results giving bounds for the number of elements representable as a subsequence sum of S.

All of the above concerns ordinary subsequence sum questions. Since the establishment of Caro's conjectured weighted Erdős-Ginzburg-Ziv Theorem [25], there has been considerable renewed interest to consider various weighted subsequence sum questions [51] [50] [49] [42] [38] [34] [33] [32] [30] [29] [23] [20] [2] [3] [4] [5] [6]. The basic idea is that given a sequence S of terms from an abelian group and a sequence W of integers (or, in the most general form, a sequence of homomorphisms between G and another abelian group G' [52]), one can instead consider which elements can be represented in the form $w_1 s_1 + \ldots + w_n s_n$ with the w_i and s_i being the terms of some subsequence from W and S, respectively. In this way, the sequence W is viewed as providing a list of potential weights, and one wishes to know which elements can be represented as a W-weighted subsequence sum rather than an ordinary subsequence sum, which is just the case when all terms in the weight sequence W are equal to 1. Formally, for a sequence $W = w_1 \cdot \ldots \cdot w_n$ of integers $w_i \in \mathbb{Z}$ and an equal length sequence $S = s_1 \cdot \ldots \cdot s_n$ with terms $s_i \in G$, we let

$$W \odot S = \{w_{\tau(1)} g_1 + \ldots + w_{\tau(n)} g_n : \; \tau \text{ a permuation of } \{1, 2, \ldots, n\}\}.$$

With this notation, the weighted Erdős-Ginzburg-Ziv Theorem says that if W is any zero-sum modulo $|G|$ sequence of integers and S is a sequence of terms from G with length $|S| \geq 2|G| - 1$, then S has a

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$|G|$-term subsequence S' with $0 \in W \odot S'$. It is still an open conjecture of Bialostocki that the weaker hypothesis $|S| = |G|$ with S zero-sum is enough to guarantee $0 \in W \odot S$ when $|G|$ is even [10] [31].

If $n = |S| \leq |W|$ and all terms of W are distinct (as will be the case in this paper), so that one may associate W with the set $A := \mathrm{supp}(W) = \{w_i : \ w_i \text{ a term of } W\}$, then

$$W \odot S = \{w_1 s_1 + \ldots + w_n s_n : \, w_i \in A, \, w_i \neq w_j \text{ for } i \neq j\}.$$

When all $s_i = 1$, then this is precisely the restricted sumset

$$A \hat{+} \ldots \hat{+} A = \{a_1 + \ldots + a_n : \, a_i \in A, \, a_i \neq a_j \text{ for } i \neq j\},$$

which has been extensively studied; see for instance [43] [35] [13] [7] [37] [44]. Thus, for such W, studying $W \odot S$ is the same as studying a particular weighted restricted sumset question. In the extreme case when $|A| = n$, there is only one possible element from the restricted sumset $A \hat{+} \ldots \hat{+} A$. However, once the s_i are allowed to take on more general values, the study of such weighted restricted sumsets $W \odot S$ quickly becomes more complicated.

Much of the initial attention regarding weighted subsequence sum problems remained on analogs of the Davenport Constant and Erdős-Ginzburg-Ziv Theorem, often providing results valid when both sequences W and S are arbitrary, the idea being that restricting such results to the case when W is the constant 1 sequence gives an extension of more classical subsequence sum questions. The weighted Erdős-Ginzburg-Ziv Theorem mentioned above gives one such example. However, there is a very natural non-constant weight sequence that has not yet been much studied: namely, one can consider W-weighted subsequence sums of S when W is an arithmetic progression of integers. The focus of this paper is to investigate such weighted subsequence sums. In particular, since the terms of W are generally all distinct, this is also a particular type of weighted restricted sumset question as discussed above.

Indeed, the main goal is to show that $|G| + 1$ is the minimal length of a sequence S from a finite abelian group G needed to guarantee that every element of G is representable as a W-weighted subsequence sum, where W is an arithmetic progression of $|S|$ consecutive integers (provided the terms of S do not all come from a coset of a proper subgroup, which is easily seen to be a necessary condition for $W \odot S = G$ to hold). Moreover, we also characterize the structure of those sequences of length one less which do not realize every element of G as a W-weighted subsequence sum and give a lower bound for $|W \odot S|$ in terms of $|S|$, which, at least in rather limited special cases, is tight (simply consider $S = 0^{|S|-1} g$ with g a generator of G). In the notation of the following section, our main result is as follows. It is worth noting that Theorem 1.1 contains, as a very special case, the main result from [31], which was devoted to proving the aforementioned conjecture of Bialostocki in the case when the weight sequence is an arithmetic progression of even difference.

Theorem 1.1. *Let G be a finite abelian group, let S be a sequence of terms from G not all contained in a coset of a proper subgroup, and let W be a sequence of $|S|$ consecutive integers.*

- $|W \odot S| \geq \min\{|G| - 1, \, |S|\}$.
- *If $|S| \geq |G| + 1$, then $W \odot S = G$. Indeed, $W' \odot S' = G$ for some subsequence $S' \mid S$ with $|S'| = |G|$, where $W' = (0)(1) \cdot \ldots \cdot (|G| - 1) \in \mathcal{F}(\mathbb{Z})$.*
- *If $|S| = |G|$ and $W \odot S \neq G$, then $|G| \geq 3$ and either*
  - (i) *$G \cong C_2 \oplus C_2$, $|\mathrm{supp}(S)| = |S| = |G| = 4$ and $W \odot S = G \setminus \{0\}$, or*
  - (ii) *G is cyclic, $(-g' + S) = 0^{|G|-2}(g)(-g)$, for some $g, \, g' \in G$ with $\mathrm{ord}(g) = |G|$, and $W \odot S = G \setminus \{\frac{1}{2}(|G| - 1)|G|g'\}$. In particular, $W \odot S$ contains every generator $h \in G$.*

In the final sections, we give simple corollaries of the above theorem first regarding whether a linear equation has a solution modulo n with all members of the solution distinct modulo n, and then concerning the pattern of multiplicities possible in a maximal length minimal zero-sum sequence over a rank 2 finite abelian group, thus providing more refined information than immediately available from the recent characterization of such sequences [16] [19] [48] [46] [9].

2. Preliminaries

Our notation and terminology are consistent with [22] [21] [17]. We briefly gather some key notions and fix the notation concerning sequences and sumsets over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, \, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. Throughout, all abelian groups will be written additively. We let C_n denote a cyclic group with n elements.

Let G be a finite abelian group, $H \leq G$ a subgroup and $A \subseteq G$ a subset. We use $\phi_H : G \to G/H$ to denote the canonical homomorphism and let $\langle A \rangle_* = \langle A - A \rangle$ denote the minimal subgroup $\langle A \rangle_*$ for which A is contained in a $\langle A \rangle_*$-coset. Note that $\langle A \rangle_* = \langle A - a \rangle$ for any $a \in A$.

For subsets $A, \, B \subseteq G$, we set

$$A + B = \{a + b : a \in A, \, b \in B\}$$

for their *sumset* and, if $B = \{b\}$, write $A + B = A + b = \{a + b : a \in A\}$. We write

$$\mathsf{H}(A) = \{g \in G : g + A = A\}$$

for the *stabilizer* of A, which is in fact a subgroup of G for finite A. If A is a union of H-cosets, for some subgroup $H \le G$, then we say A is H*-periodic*, which is equivalent to saying $H \le \mathsf{H}(A)$, i.e, that $A + H = A$. We call A *periodic* if $\mathsf{H}(A)$ contains a nontrivial subgroup, and otherwise A is *aperiodic.* An element $x \in (A + H) \setminus A$ is referred to as an H*-hole* of A.

We use $\mathcal{F}(G)$ to denote all finite length (unordered) sequences with terms from G, refer to the elements of $\mathcal{F}(G)$ simply as sequences, and write all such sequences multiplicatively, so that a sequence $S \in \mathcal{F}(G)$ is written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all } g \in G.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S and say that S contains g if $\mathsf{v}_g(S) > 0$. The notation $S_1 \mid S$ indicates that S_1 is a subsequence of S, that is, $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. A sequence of finite, nonempty subsets of G is called a *setpartition.*

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$$

and $n \in \mathbb{N}$, we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S,$$

$$\Sigma_n(S) = \left\{ \sum_{i \in I} g_i : \; I \subseteq [1, l],\, |I| = n \right\} \subseteq G \qquad \text{the } \textit{set of } n\textit{-term subsequence sums} \text{ of } S,$$

$$\mathrm{supp}(S) = \{g_1, \ldots, g_l\} = \{g \in G : \mathsf{v}_g(S) > 0\} \qquad \text{the } \textit{support} \text{ of } S, \text{ and}$$

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) : g \in G\} \qquad \text{the } \textit{maximum multiplicity} \text{ of a term of } S.$$

For $g' \in G$, we write

$$(g' + S) = (g' + g_1) \cdot \ldots \cdot (g' + g_l) = \prod_{g \in G} (g' + g)^{\mathsf{v}_g(S)} = \prod_{g \in G} g^{\mathsf{v}_{g - g'}(S)} \in \mathcal{F}(G).$$

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no non-trivial zero-sum subsequence, and
- a *minimal zero-sum sequence* if $|S| > 0$, $\sigma(S) = 0$, and every subsequence $S' \mid S$ with $0 < |S'| < |S|$ is zero-sum free.

The Davenport constant $\mathsf{D}(G)$ of G is then the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge l$ has a non-trivial zero-sum subsequence (equivalently, S is not zero-sum free).

The following is one of the foundational results of set addition. Note that multiplying both sides of the inequality from Kneser's Theorem [36] [39] [22] by $|H|$ yields

$$|\sum_{i=1}^{n} A_i| \ge \sum_{i=1}^{n} |A_i + H| - (n-1)|H| = \sum_{i=1}^{n} |A_i| - (n-1)|H| + \rho,$$

where $\rho := \sum_{i=1}^{n} |(A_i + H) \setminus A_i|$ is the number of H-holes in the sets A_i. Additionally, if $\sum_{i=1}^{n} A_i$ is aperiodic, then Kneser's Theorem implies

$$|\sum_{i=1}^{n} A_i| \ge \sum_{i=1}^{n} |A_i| - n + 1.$$

Theorem 2.1 (Kneser's Theorem)**.** *Let G be an abelian group, let $A_1, \ldots, A_n \subseteq G$ be finite, nonempty subsets, and let $H = \mathsf{H}(\sum_{i=1}^{n} A_i)$. Then*

$$|\sum_{i=1}^{n} \phi_H(A_i)| \ge \sum_{i=1}^{n} |\phi_H(A)| - n + 1.$$

We will also need the following simple consequence of the Pigeonhole Principle [39].

Lemma 2.2. *Let G be a finite abelian group and let $A,\, B \subseteq G$ be nonempty subsets. If $|A|+|B|-1 \geq |G|$, then $A+B=G$.*

3. Proof of Theorem 1.1

For two sequences $W \in \mathcal{F}(\mathbb{Z})$ and $S \in \mathcal{F}(G)$, where G is an abelian group, set

$$W \odot S = \{w_1 g_1 + \ldots + w_r g_r :\, w_1 \cdot \ldots \cdot w_r \mid W,\; g_1 \cdot \ldots \cdot g_r \mid S \;\text{ and }\; r = \min\{|W|,\, |S|\}\}.$$

Note that

$$W \odot S = (W0^{|S|-r}) \odot (S0^{|W|-r}) \quad \text{ with } \quad |W0^{|S|-r}| = |S0^{|W|-r}| = \max\{|W|,\, |S|\},$$

where $r = \min\{|W|,\, |S|\}$. Also, if $|W| \geq |S|$, then

$$(W+w) \odot S = W \odot S + w\sigma(S) \quad \text{ for all } \; w \in \mathbb{Z}, \tag{3.1}$$

while if $|S| \geq |W|$, then

$$W \odot (S+g) = W \odot S + \sigma(W)g \quad \text{ for all } \; g \in G. \tag{3.2}$$

In particular, if $|W| = |S|$, then $G = W \odot S$ if and only if $G = (W+w) \odot (S+g)$ for all $w \in \mathbb{Z}$ and $g \in G$.

We begin with a lemma dealing with the case $|S| = 3$ for Theorem 1.1.

Lemma 3.1. *Let G be an abelian group, let $W = (0)(1) \cdot \ldots \cdot (|W|-1) \in \mathcal{F}(\mathbb{Z})$ be a sequence of consecutive integers, let $x,\, y \in G \setminus \{0\}$ be nonzero elements with $\langle x,\, y\rangle = G$, and set $S = xy \in \mathcal{F}(G)$.*

(i) *If $|W| \geq 3$, then $\langle W \odot S\rangle_* = G$.*
(ii) *If $x = y$, then $|W \odot S| \geq \min\{|G|,\, 2|W|-3\}$.*
(iii) *If $x \neq y$, then $|W \odot S| \geq \min\{|G|-1,\, 2|W|-2\}$.*

Proof. If $|W| \leq 2$, then the lemma is easily verified. So we may assume $|W| \geq 3$. In this case, $x,\, 2x,\, 2x+y \in W \odot S$, so that

$$\langle W \odot S\rangle_* \supseteq \langle x, 2x, 2x+y\rangle_* = \langle x, x+y\rangle = \langle x, y\rangle = G,$$

whence $\langle W \odot S\rangle_* = G$ follows, yielding (i). If $x = y$, then

$$W \odot S = \{x+0, 2x+0, \ldots, (|W|-1)x+0, (|W|-1)x+x, \ldots, (|W|-1)x+(|W|-2)x\},$$

from which (ii) is readily deduced. Therefore it remains to prove the lower bound for $|W \odot S|$ when $x \neq y$.

Without loss of generality, assume $\mathrm{ord}(x) \geq \mathrm{ord}(y)$. Let $r = |W| \geq 3$ and set $H = \langle x\rangle$. Since $G/H = \langle \phi_H(y)\rangle$, it follows that

$$|H| = \mathrm{ord}(x) \geq \mathrm{ord}(y) \geq \mathrm{ord}(\phi_H(y)) = |G/H|.$$

Now we have

$$W \odot S = \left\{ \begin{array}{lllll} \square & 0+y & 0+2y & \cdots & 0+(r-1)y \\ x & \square & x+2y & \cdots & x+(r-1)y \\ 2x & 2x+y & \square & \cdots & 2x+(r-1)y \\ 3x & 3x+y & 3x+2y & \cdots & 3x+(r-1)y \\ \vdots & \vdots & \vdots & & \vdots \\ (r-1)x & (r-1)x+y & (r-1)x+2y & \cdots & \square \end{array} \right\}. \tag{3.3}$$

Note that each column consists of elements from the same H-coset. We divide the remainder of the proof into several cases based off the number of H-cosets in G.

Case 1: $|G/H| \geq 3$. If $r \leq |G/H| \leq |H| = \mathrm{ord}(x)$, then all columns in (3.3) correspond to distinct H-cosets filled with distinct elements, whence $|W \odot S| = r(r-1) \geq 2r-2$. If $|G/H|+1 \leq r \leq |H| = \mathrm{ord}(x)$, then the first $|G/H|$ columns in (3.3) are distinct and each contain at least $r-1$ elements, whence $|W \odot S| \geq (r-1)|G/H| \geq 3r-3 \geq 2r-2$. Finally, it remains to consider the case $r > |H| = \mathrm{ord}(x)$, for which $\mathrm{ord}(x) = |H|$ must be finite. Let $r = |H|+s$ with $s \geq 1$. In this case, we see that the first $|G/H|$ columns cover all distinct H-cosets and are each missing at most one element, while the first s of these columns are missing no elements. In consequence, $|W \odot S| \geq (|H|-1)|G/H| + \min\{|G/H|,\, s\}$. If $s \geq |G/H|$, then $|W \odot S| \geq |G|$ follows, as desired. Otherwise, when $1 \leq s \leq |G/H|-1 \leq |H|-1$, we can recall that $r = |H|+s$ and $|G/H| \geq 3$ and thus conclude that

$$\begin{aligned} |W \odot S| &\geq (|H|-1)|G/H| + s \geq 3|H|-3+s = r-2+|H|+(|H|-1) \\ &\geq r-2+|H|+s = 2r-2, \end{aligned}$$

also as desired.

Case 2: $|G/H| = 2$. In this case, since $r \geq 3 > |G/H|$, we see that the first two columns of (3.3) cover both distinct H-cosets. If $r \leq \mathrm{ord}(x) = |H|$, then there are $r-1$ elements in both these columns, whence $|W \odot S| \geq 2(r-1)$, as desired. On the other hand, if $r \geq \mathrm{ord}(x)+1$, then the first column is missing no element while the second column is missing at most one, whence $|W \odot S| \geq |G|-1$, also as desired.

Case 3: $|G/H| = 1$. In this case, x generates G, and thus $y = \alpha x$ for some $\alpha \in \mathbb{Z}$ with $\alpha \in (-\frac{n}{2}, \lfloor \frac{n+1}{2} \rfloor]$, where $n := \mathrm{ord}(x) = |G|$. It suffices to prove (iii) when

$$|W| = r \leq \left\lceil \frac{n+1}{2} \right\rceil,$$

as for larger $|W|$, one can simply apply (iii) using $r = \lceil \frac{n+1}{2} \rceil$ and note that $2r-2 \geq n-1 = |G|-1$ holds in this case. Thus, in view of $r \geq 3$, it follows that $n = |G| \geq 4$. To simplify notation, we may assume $x = 1$ generates the cyclic group $G \cong C_n$.

Now, from (3.3), we know that $\{1, 2, \ldots, (r-1)\} \subseteq W \odot S$. We also have

$$\{0+\alpha,\ 2+\alpha,\ 3+\alpha,\ \ldots, (r-1)+\alpha\} \subseteq W \odot S. \tag{3.4}$$

Note that $r-1+\alpha \leq \lceil \frac{n+1}{2} \rceil - 1 + \lfloor \frac{n+1}{2} \rfloor = n$. Thus, if $\alpha \geq r$, then the elements from (3.4) will be disjoint from $\{1, 2, \ldots, (r-1)\} \subseteq W \odot S$, whence $|W \odot S| \geq 2(r-1)$, as desired. Likewise, if $\alpha \leq -(r-1)$, then we have $n+\alpha \geq \frac{n+1}{2} > r-1$, and the elements in (3.4) will again be disjoint from $\{1, 2, \ldots, (r-1)\} \subseteq W \odot S$, yielding the desired bound $|W \odot S| \geq 2(r-1)$ once more. Thus, in both cases, (iii) holds, and we may now assume

$$-r+2 \leq \alpha \leq r-1. \tag{3.5}$$

Suppose $\alpha \geq 0$. Then, in view of $y \neq x$, $y \neq 0$ and (3.5), we have $\alpha \in [2, r-1]$. The sums $0+1, 0+2, \ldots, 0+r-1 \in W \odot S$ show that $[1, r-1] \subseteq W \odot S$. The sums $j \cdot \alpha + (r-\alpha+i)$, for $j \in [1, r-1]$ and $i \in [0, \alpha-1] \setminus \{j-r+\alpha\}$, show that each interval $[r+(j-1)\alpha, r+j\alpha-1]$ is contained in $W \odot S$ apart from possibly the element $j\alpha + (r-\alpha+i) = j\alpha + j$ when $i = j-r+\alpha \in [0, \alpha-1]$, for $j \in [1, r-1]$. In particular, in order for an element to be missing from the interval $[r+(j-1)\alpha, r+j\alpha-1]$ in $W \odot S$, we must have $j-r+\alpha \geq 0$, i.e., $j \geq r-\alpha$. As a result, we conclude from all of the above that

$$[1, (r-\alpha+1)\alpha + r-\alpha] \setminus \{(r-\alpha)\alpha + (r-\alpha)\} \subseteq W \odot S,$$

from which, in view of $\alpha \in [2, r-1]$ and $r \geq 3$, it is easily deduced that

$$|W \odot S| \geq \min\{|G|-1,\ (r-\alpha+1)\alpha + r-\alpha-1\} \tag{3.6}$$

$$\geq \min\{|G|-1,\ 3r-5,\ 2r-2\} \geq \min\{|G|-1,\ 2r-2\}, \tag{3.7}$$

as desired. So we now assume $\alpha < 0$.

Since $\alpha < 0$, we infer from (3.5) that $\alpha \in [-r+2, -1]$. Furthermore, (3.5) also gives

$$r \geq |\alpha| + 2. \tag{3.8}$$

If $\alpha = -1$, then we clearly have

$$[-(r-1), -1] \cup [1, r-1] = ([1, r-1] \odot (-1) + 0 \cdot 1) \cup (0 \cdot (-1) + [1, r-1] \odot 1) \subseteq W \odot S,$$

from which (iii) easily follows. If $\alpha = -2$, then $[1, r-1] = 0 \cdot (-2) + [1, r-1] \odot 1 \subseteq W \odot S$ and

$$\begin{aligned}&\{1 \cdot (-2) + 2 = 0, \quad 1 \cdot (-2) + 0 = -2,\\ &2 \cdot (-2) + 1 = -3, \quad 2 \cdot (-2) + 0 = -4, \quad 3 \cdot (-2) + 1 = -5, \quad 3 \cdot (-2) + 0 = -6, \quad \ldots,\\ &(r-1) \cdot (-2) + 1 = -2r+3, \quad (r-1) \cdot (-2) + 0 = -2r+2\} \subseteq W \odot S.\end{aligned}$$

Consequently,

$$[-2r+2, r-1] \setminus \{-1\} \subseteq W \odot S,$$

from which it is easily deduced that $|W \odot S| \geq \min\{|G|-1,\ 3r-3\} \geq \min\{|G|-1,\ 2r-2\}$, as desired. Therefore we may assume $\alpha \leq -3$, in which case (3.8) gives

$$r \geq |\alpha| + 2 \geq 5.$$

We know $[1, r-1] = 0 \cdot \alpha + [1, r-1] \odot 1 \subseteq W \odot S$. Since $\alpha \leq -3$ and $3 \leq |\alpha| \leq r-2$, we also have $1 \cdot \alpha + |\alpha| \cdot 1 = 0 \in W \odot S$, whence

$$[0, r-1] \subseteq W \odot S.$$

Next we claim that, for each $j \in [1, r-1]$, $W \odot S$ also contains all elements from $[j\alpha, (j-1)\alpha - 1]$ except possibly $j\alpha + j$. Indeed, to see this, we have only to note that $j \cdot \alpha + \beta \cdot 1 \in W \odot S$ for $\beta \in [0, |\alpha|-1] \setminus \{j\}$. Next, since $\alpha \leq -2$, it follows that

$$j\alpha + j = (j+1) \cdot \alpha + (|\alpha| + j) \cdot 1 \in W \odot S \quad \text{ for } j \leq r-1-|\alpha|.$$

As a result, we conclude from the above work that

$$[(r-|\alpha|+1)\alpha+(r-|\alpha|+1)+1, r-1]\setminus\{(r-|\alpha|)\alpha+(r-|\alpha|)\}\subseteq W\odot S,$$

which, combined with $|\alpha|\in[3,r-2]$ and $r\geq 5$, allows us to easily infer that

$$\begin{aligned}|W\odot S| &\geq \min\{|G|-1,\ (r-|\alpha|+2)|\alpha|-3\}\\ &\geq \min\{|G|-1,\ 3r-6,\ 4r-11\}\geq \min\{|G|-1,\ 2r-2\},\end{aligned}$$

completing the proof. □

We will need the following technical refinement of the case $|W|=3$ from Lemma 3.1.

Lemma 3.2. *Let G be an abelian group with $|G|\geq 5$, let $W=(0)(1)(2)\in\mathcal{F}(\mathbb{Z})$ be a sequence of* 3 *consecutive integers, let $x,\ y,\ z\in G$ be distinct elements with $\langle x,\ y,\ z\rangle_*=G$, and set $S=xyz\in\mathcal{F}(G)$. Suppose* $\mathrm{ord}(x-z)$, $\mathrm{ord}(y-z)$, $\mathrm{ord}(x-y)\geq 3$. *Then there exists a subset $X\subseteq W\odot S$ with $|X|=4$, $|X\cap(3z+\langle x,\ z\rangle_*)|\geq 2$ and $\langle X\rangle_*=G$. Furthermore, if $G\ncong C_6$, then $|\mathsf{H}(X)|\neq 2$.*

Proof. In view of (3.2), we can w.l.o.g. translate S so that $z=0$. If the three terms of S are in arithmetic progression, say $S=0(x)(2x)$, $S=0(y)(2y)$ or $S=(-x)0(x)$, then $W\odot S=\{1,2,4,5\}\odot x$, $W\odot S=\{1,2,4,5\}\odot y$ or $W\odot S=\{-2,-1,1,2\}\odot x$, and the lemma is easily verified taking $X=W\odot S$. Therefore we may assume S is not in arithmetic progression, whence

$$y\notin\{-x,0,x,2x\}\qquad\text{and}\qquad x\notin\{-y,0,y,2y\}. \tag{3.9}$$

Consider the set $X:=\{x,2x,2x+y,y\}\subseteq W\odot S$. In view of (3.9) and $\mathrm{ord}(x)\geq 3$, we have $|X|=4$. We also have $\langle x,2x,2x+y\rangle_*=\langle x,x+y\rangle=\langle x,y\rangle=\langle x,y,z=0\rangle_*=G$, so that $\langle X\rangle_*=G$. Clearly, $|X\cap\langle x\rangle|\geq 2$.

Finally, if $|\mathsf{H}(X)|=2$, then there must be a pairing up of the 4 elements of X such that the difference of elements in each pairing is equal to the same order two element. There are three such possible pairings: $\{x,2x\}$ and $\{y,2x+y\}$; $\{x,y\}$ and $\{2x,2x+y\}$; $\{x,2x+y\}$ and $\{y,2x\}$. Since $\mathrm{ord}(x)\geq 3$ and $\mathrm{ord}(y)\geq 3$, we cannot have x and $2x$, nor $2x$ and $2x+y$, being in the same cardinality two coset, which rules out the first two possible pairings. On the other hand, if $\{x,2x+y\}$ and $\{y,2x\}$ are both cosets of the same order 2 subgroup, then we must have $x+y=(2x+y)-x=2x-y$, contradicting (3.9). As this exhausts all possible pairings, we conclude that $|\mathsf{H}(X)|=2$ does not hold, completing the proof. □

Next, we show that if the terms of S generate G (up to translation), then so do the elements of $W\odot S$.

Lemma 3.3. *Let G be an abelian group, let $S\in\mathcal{F}(G)$ be a sequence, and let $W\in\mathcal{F}(\mathbb{Z})$ be a sequence of consecutive integers. If $|W|=|S|$, then $\langle W\odot S\rangle_*=\langle\mathrm{supp}(S)\rangle_*$.*

Proof. In view of (3.2), (3.1) and $|W|=|S|$, there is no loss in generality if we translate W and S such that $W=(0)(1)\cdot\ldots\cdot(|S|-1)$ and $0\in\mathrm{supp}(S)$. If $|S|\leq 2$, then the lemma is easily verified. We proceed by induction on $|S|$. If $\mathrm{supp}(S)=\{0\}$, then $\langle\mathrm{supp}(S)\rangle_*=\{0\}=\langle W\odot S\rangle_*$. Therefore we may assume $|\mathrm{supp}(S)|\geq 2$. We trivially have $\langle W\odot S\rangle_*\subseteq\langle\mathrm{supp}(S)\rangle=\langle\mathrm{supp}(S)\rangle_*$, with the latter equality in view of $0\in\mathrm{supp}(S)$. Therefore, it suffices to show the reverse inclusion $\langle\mathrm{supp}(S)\rangle_*\subseteq\langle W\odot S\rangle_*$.

Let $x\in\mathrm{supp}(S)$ be nonzero. Let $K:=\langle\mathrm{supp}(Sx^{-1})\rangle$. Since $0\in\mathrm{supp}(Sx^{-1})$, we have

$$K=\langle\mathrm{supp}(Sx^{-1})\rangle_*=\langle\mathrm{supp}(Sx^{-1})\rangle.$$

Thus, by induction hypothesis, we conclude that

$$\langle(0\cdot x)+\big(W0^{-1}\odot Sx^{-1}\big)\rangle_*=K;$$

moreover, since $R\odot(Sx^{-1})\subseteq\langle\mathrm{supp}(Sx^{-1})\rangle=K$ for any sequence of integers $R\in\mathcal{F}(\mathbb{Z})$, we actually have

$$(0\cdot x)+\big(W0^{-1}\odot Sx^{-1}\big)\subseteq K.$$

Consequently, to show $\langle\mathrm{supp}(S)\rangle_*\subseteq\langle W\odot S\rangle_*$, it suffices to show that $W\odot S$ contains some element from $x+K$. However, clearly

$$(1\cdot x)+\big((0)(2)(3)\cdot\ldots\cdot(|S|-1)\odot Sx^{-1}\big)\subseteq W\odot S$$

is a nontrivial subset of $x+\langle\mathrm{supp}(Sx^{-1})\rangle=x+K$, so that $W\odot S$ indeed contains some element from $x+K$, completing the proof. □

The following lemma can be found in [26] as observation (c.5). See [28, Proposition 5.2] for a more detailed proof.

Lemma 3.4. *Let G be an abelian group, let $A\subseteq G$ be a finite, nonempty subset, and let $x\in G\setminus A$. If $A\cup\{x\}$ is H-periodic with $|H|\geq 3$, then $A\cup\{y\}$ is aperiodic for every $y\in G\setminus\{x\}$.*

We now proceed with the proof of our main result.

Proof of Theorem 1.1. In view of (3.2) and (3.1), our problem is invariant when translating S or W, so we may w.l.o.g. assume $0 \in \mathrm{supp}(S)$ is a term with maximum multiplicity $\mathsf{v}_0(S) = \mathsf{h}(S)$. For $|G| \leq 4$, the theorem is quickly verified by an exhaustive enumeration of all possible sequences. Likewise when $|S| \leq 2$, while the case $|S| = 3$ follows from Lemma 3.1(ii)–(iii). Therefore we may assume

$$|G| \geq 5 \quad \text{ and } \quad |S| \geq 4$$

and proceed by a double induction on $(|G|, |S|)$, assuming the theorem proved for any sequence over a smaller cardinality subgroup as well as any sequence over G with smaller length than S.

In view of (3.2), we see that if $(-g' + S) = 0^{|G|-2}(g)(-g)$, for some $g,\, g' \in G$ with $\mathrm{ord}(g) = |G|$, then $W \odot S = G \setminus \{\frac{1}{2}(|G|-1)|G|g'\}$; in particular, $W \odot S$ contains every generator $h \in G$ in view of $|G| \geq 3$. Thus the latter conclusions of (ii) are simple consequences of the structural characterization of S given there.

Next let us show that the structural characterization from the third part of the theorem implies the second part of the theorem. Indeed, if $|S| = |G| + 1$ and $W' \odot S0^{-1} \neq G$, then recalling that $|G| \geq 5$ and applying the characterization to $S0^{-1}$ yields $S = g'^{|G|-2}(g'+g)(g'-g)0$ for some $g,\, g' \in G$ with $\mathrm{ord}(g) = |G|$. Since $\mathrm{ord}(g) = |G| \geq 5$, we have $(g' + g) \neq (g' - g)$. Thus, if $g' \neq 0$, then $|G| - 2 \leq \mathsf{h}(S) = \mathsf{v}_0(S) \leq 2$, contradicting that $|G| \geq 5$. Therefore we conclude that $S = 0^{|G|-1}(g)(-g)$ with $\mathrm{ord}(g) = |G|$, and now clearly the subsequence $S' = 0^{|G|-1}g$ has $W' \odot S' = G$. So we see that it suffices to prove the first and third parts of the theorem. In particular, we can assume $|S| \leq |G|$ and we need to show either $|W \odot S| \geq |S|$ or else $|S| = |G|$ with S being described by (ii).

Case 1: $|\,\mathrm{supp}(S)| = 2$.

In this case, in view of $\langle \mathrm{supp}(S) \rangle = G$, we have $S = 0^{|S|-\alpha}g^{\alpha}$ with $\mathrm{ord}(g) = |G|$ and $1 \leq \alpha \leq |S| - 1 \leq |G| - 1$. As a result, it is easily seen that $W \odot S$ is an arithmetic progression with difference g and length

$$|W \odot S| = \min\{|G|,\, |\Sigma_{\alpha}([0, |S|-1])|\} = \min\{|G|,\, \alpha|S| - \alpha^2 + 1\} \geq |S|,$$

where the final equality follows in view of $1 \leq \alpha \leq |S| - 1 \leq |G| - 1$. Thus $|W \odot S| \geq |S|$, as desired. This completes Case 1.

Case 2: $\mathsf{h}(S) \geq |S| - 2$.

Since $\langle \mathrm{supp}(S) \rangle = G$ with $|G| \geq 5$, we trivially have $\mathsf{h}(S) \leq |S| - 1$. If $\mathsf{h}(S) = |S| - 1$, then $\langle \mathrm{supp}(S) \rangle = G$ and $\mathsf{v}_0(S) = \mathsf{h}(S)$ ensure that $S = 0^{|S|-1}g$ with $\mathrm{ord}(g) = |G|$, and now Case 1 completes the proof. So it remains to consider $\mathsf{h}(S) = |S| - 2$ for Case 2. In this case, $S = 0^{|S|-2}xy$ with $x,\, y \in G \setminus \{0\}$. In view of Case 1, we may assume $x \neq y$. Note

$$(W(|S|-1)^{-1} \odot T) \cup (W0^{-1} \odot T) \subseteq W \odot S, \tag{3.10}$$

where $T := xy \in \mathcal{F}(G)$. Lemma 3.1(iii) and $|S| \geq 4$ together imply that

$$|(W(|S|-1)^{-1}) \odot T| \geq \min\{|G| - 1,\, 2|W| - 4\} = \min\{|G| - 1,\, 2|S| - 4\} = \min\{|G| - 1,\, |S|\}.$$

In consequence, if $|S| \leq |G| - 1$, then the proof is complete, so we assume $|S| = |G|$. In this case, we have

$$|W(|S|-1)^{-1} \odot T| \geq |G| - 1$$

and likewise $|W0^{-1} \odot T| \geq |G| - 1$. Combined with (3.10), we once more obtain the desired conclusion $W \odot S = G$ unless $W0^{-1} \odot T = W(|S|-1)^{-1} \odot T$ with $|W0^{-1} \odot T| = |G| - 1$. In particular, $W0^{-1} \odot T$ is aperiodic, in which case (3.1) shows that $W0^{-1} \odot T = W(|S|-1)^{-1} \odot T$ is only possible if $\sigma(T) = x + y = 0$. Thus $y = -x$. We now know $S = 0^{|G|-2}x(-x)$. Hence, since $\langle \mathrm{supp}(S) \rangle = G$, we conclude that x generates G, whence G is cyclic with $\mathrm{ord}(x) = |G|$, which gives the desired conclusion of (ii). This completes Case 2.

Case 3: There exists a subsequence $T \mid S$ with $\langle \mathrm{supp}(T) \rangle_* = H$, where $H < G$ is a proper, nontrivial subgroup, and either $|T| \geq |H| + 1$ (if $|H| \geq 3$) or $|T| \geq |H|$ (if $|H| = 2$).

Let $W_T = (0)(1) \cdot \ldots \cdot (|H| - 1) \in \mathcal{F}(\mathbb{Z})$. By induction hypothesis, we can apply the theorem to T to conclude that $W_T \odot T'$ is an H-coset for some subsequence $T' \mid T$ with $|T'| = |H|$. By translating appropriately, we can w.l.o.g. assume $0 \in \mathrm{supp}(T')$, though we may lose that $\mathsf{h}(S) = \mathsf{v}_0(S)$. Let

$$\langle \mathrm{supp}(\phi_H(ST'^{-1})) \rangle_* = K/H, \quad \text{ where } \; H \leq K \leq G.$$

Then all terms of $\phi_H(ST'^{-1})$ are contained in a single K/H-coset, say $\mathrm{supp}\{\phi_K(ST'^{-1})\} = \{\phi_K(\alpha)\}$, where $\alpha \in G$. Consequently, since $\langle \mathrm{supp}(S) \rangle_* = \langle \mathrm{supp}(S) \rangle = G$, so that $\langle \mathrm{supp}(\phi_K(S)) \rangle = G/K$, and since $\mathrm{supp}(T') \subseteq H \subseteq K$, so that $\mathrm{supp}(\phi_K(T')) = \{0\}$, it follows that

$$\langle \phi_K(\alpha) \rangle = G/K. \tag{3.11}$$

If $T \neq T'$, which holds whenever $|H| \geq 3$, then it follows in view of $\mathrm{supp}(\phi_H(T)) = \{0\}$ that $\mathrm{supp}(\phi_H(ST'^{-1})) = \mathrm{supp}(\phi_H(S))$, whence $\langle \mathrm{supp}(\phi_H(ST'^{-1}))\rangle_* = \langle \mathrm{supp}(\phi_H(S))\rangle_* = G/H$. In summary,

$$K = G \quad \text{when } T' \neq T \text{ or } |H| \geq 3. \tag{3.12}$$

Next, let us show that

$$|W \odot S| \geq 2|H|. \tag{3.13}$$

If $|\mathrm{supp}(\phi_H(ST'^{-1}))| \geq 2$, then $|WW_T^{-1} \odot \phi_H(ST'^{-1})| \geq 2$, which combined with the fact that $W_T \odot T'$ is an H-coset yields (3.13). Therefore assume instead $\mathrm{supp}(\phi_H(ST'^{-1})) = \{\phi_H(\beta)\}$, where $\beta \in \mathrm{supp}(ST'^{-1})$. Since $\mathrm{supp}(T) \subseteq H$, if $\phi_H(\beta) = 0$, then $\mathrm{supp}(S) \subseteq H < G$ follows, contradicting that $\langle \mathrm{supp}(S)\rangle = G$. Therefore $\phi_H(\beta) \neq 0$. However, if $|H| \geq 3$, then ST'^{-1} contains a term from T, and thus a term from H, in which case $\phi_H(\beta) = 0$, contrary to what we just noted. Therefore we can now assume $|H| = |T| = 2$ for proving (3.13). Now $(x + W_T) \odot T' = H$ for all $x \in [0, |S| - 2]$. Thus, if (3.13) fails, then we must have

$$\Big|\bigcup_{x \in [0, |S|-2]} W(x + W_T)^{-1} \odot \phi_H(\beta)^{|S|-2}\Big| = 1. \tag{3.14}$$

As a result, since $|S| \geq 4$, comparing the values $x = 0$ and $x = 1$ in (3.14) shows that

$$\Big(\frac{(|S|-1)|S|}{2} - 1\Big)\phi_H(\beta) = \Big(\frac{(|S|-1)|S|}{2} - 3\Big)\phi_H(\beta),$$

whence $2\phi_H(\beta) = 0$. However, since $\mathrm{supp}(\phi_H(S)) = \{0, \phi_H(\beta)\}$ must generate G/H, this implies that $|G| = |G/H| \cdot |H| = 2 \cdot 2 = 4$, contradicting the assumption $|G| \geq 5$. Thus (3.13) is established in all cases.

We can assume

$$2 \leq |H| \leq \frac{|S| - 1}{2}, \tag{3.15}$$

else the desired conclusion $|W \odot S| \geq |S|$ follows from (3.13). We divide the remainder of the case into several subcases.

Subcase 3.1: $K = G$ and $|S| \geq |H| + |G/H| + 1$.

In this case, we can apply the induction hypothesis to $\phi_H(ST'^{-1})$ to conclude that

$$(WW_T^{-1}) \odot \phi_H(ST'^{-1}) = G/H.$$

Hence, since $W_T \odot T'$ is an H-coset, it follows that $G = (WW_T^{-1}) \odot (ST'^{-1}) + W_T \odot T' \subseteq W \odot S$, as desired.

Subcase 3.2: $|S| \leq |H| + |K/H| - 1 + \epsilon$, where $\epsilon = 0$ if $|K/H| \geq 3$ and $\epsilon = 1$ if $|K/H| \leq 2$.

In this case, we can apply the induction hypothesis to $WW_T^{-1} \odot \phi_H(ST'^{-1})$, recall that $W_T \odot T'$ is an H-coset, and use the bounds given by (3.15) to conclude that

$$|W \odot S| \geq |H|(|S| - |T'|) = |H|(|S| - |H|) \geq \min\{2|S| - 4,\ \frac{|S|^2 - 1}{4}\}. \tag{3.16}$$

If the theorem fails for S, then $|W \odot S| \leq |S| - 1$, which combined with (3.16) yields the contradiction $|S| \leq 3$.

Subcase 3.3: $|S| = |H| + |K/H|$.

In view of Subcase 3.2, we can assume $|K/H| \geq 3$, whence $|K| \geq 3|H| \geq 6$. Applying the induction hypothesis to $WW_T^{-1} \odot \phi_H(ST'^{-1})$ and recalling that $W_T \odot T'$ is an H-coset, we conclude that

$$|W \odot S| \geq |H|(|K/H| - 1) = |K| - |H|. \tag{3.17}$$

If the theorem fails for S, then $|W \odot S| \leq |S| - 1 = |H| + |K/H| - 1$, which combined with (3.17) yields

$$|K| \leq 2|H| + |K/H| - 1.$$

However, in view of $2 \leq |H| \leq \frac{|K|}{3}$, the above is only possible if $|K| = 6$ and $|H| = 2$. In this case, equality must hold in (3.17), which is only possible (in view of $|K/H| = 3$ and the characterization given by (ii)) if the 3 terms of $\phi_H(ST'^{-1})$ are the 3 distinct elements of some cardinality 3 coset $\phi_H(\beta) + K/H$, where $\beta \in G$. Let $K/H = \{0, \phi_H(g), 2\phi_H(g)\}$, where $\mathrm{ord}(\phi_H(g)) = 3$ and $g \in G$, so that

$$\phi_H(ST'^{-1}) = \phi_H(\beta)\phi_H(\beta + g)\phi_H(\beta + 2g).$$

Since $3 \equiv 1 \mod 2$, we have $(0)(3) \odot T' = H$, while

$$(1)(2)(4) \odot \phi_H(ST'^{-1}) = (1)(2)(4) \odot \phi_H(\beta)\phi_H(\beta+g)\phi_H(\beta+2g) = 7\phi_H(\beta) + \{0, \phi_H(g), 2\phi_H(g)\}$$

is a full K/H-coset, whence

$$7\beta + K = (0)(3) \odot T' + (1)(2)(4) \odot ST'^{-1} \subseteq W \odot S.$$

Thus $|W \odot S| \geq |K| = 6 > |S|$, as desired, which completes the subcase.

Observe that Subcases 3.1–3.3 cover all possibilities when $K = G$. Thus it remains to consider the case when $K < G$ is proper, in which case (3.12) shows $|H| = 2$. Note that the following subcase covers all remaining possibilities.

Subcase 3.4: $K < G$ is proper and $|S| \geq |H| + |K/H| + 1 = |K/H| + 3$.

In view of (3.12), we conclude there must be precisely 2 terms of S from H for this subcase, else $T \neq T'$ and $K = G$ follows, contrary to subcase hypothesis.

Suppose $|S| \geq |H| + 2|K/H| + 1 = |K| + 3$. Then $|ST'^{-1}| \geq 2|K/H| + 1 = |K| + 1 \geq 3$. Recall that all terms of ST'^{-1} are from the K-coset $\alpha + K$. Thus $\langle \text{supp}(ST'^{-1}) \rangle_* \leq K < G$. Hence, if $\langle \text{supp}(ST'^{-1}) \rangle_*$ is nontrivial, then, in view of $|ST'^{-1}| \geq |K| + 1 \geq 3$, we see that the hypotheses of Case 3 but not Subcase 3.4 hold using ST'^{-1} and $\langle \text{supp}(ST'^{-1}) \rangle_*$ in place of T and H, whence one of the previous subcases can be applied to complete the case. On the other hand, if $\langle \text{supp}(ST'^{-1}) \rangle_*$ is trivial, say w.l.o.g. $ST'^{-1} = \alpha^{|S|-2}$, then we can translate S so that $S = 0^{|S|-2}xy$ and apply Case 2 to complete the subcase. So we may instead assume

$$|S| \leq |K| + 2. \tag{3.18}$$

Since $|ST'^{-1}| = |S| - |H| \geq |K/H| + 1$ holds by hypothesis, we can apply the induction hypothesis to $WW_T^{-1} \odot \phi_H(ST'^{-1})$ and recall that $W_T \odot T'$ is an H-coset to thereby conclude that

$$|W \odot S| \geq |K|. \tag{3.19}$$

If the theorem fails for S, then we must have $|W \odot S| \leq |S| - 1$, which, in view of (3.18) and (3.19), is only possible if

$$2|K/H| + 1 = |K| + 1 \leq |S| \leq |K| + 2. \tag{3.20}$$

From (3.15), we have $|S| \geq 2|H| + 1$, which combined with (3.20) implies that $|K/H| \geq 2$.

Recall that $\text{supp}(ST'^{-1}) \subseteq \alpha + K$. Since $|K/H| \geq 2$, we infer from (3.20) that $|\phi_H(ST'^{-1})| \geq |K/H| + 1$, whence applying the induction hypothesis to $\phi_H(ST'^{-1})$ shows that there exists a subsequence $R \mid ST'^{-1}$ with $|R| = |K/H|$ such that $W' \odot \phi_H(R)$ is a K/H-coset for any sequence W' consisting of $|K/H|$ consecutive integers.

Recall that $|K| \geq |H| \geq 2$. Thus, if $|W \odot S| \geq 2|K|$, then combining this with (3.18) shows that $|W \odot S| \geq |S|$, as desired. Therefore we conclude that

$$|W \odot S| < 2|K|. \tag{3.21}$$

In view of the subcase hypothesis, $ST'^{-1}R^{-1}$ is a nonempty sequence, so we may find some $g \in \text{supp}(ST'^{-1}R^{-1})$. Since $(0)(1) \odot T' = H$ and $(2)(3) \cdot \ldots (|K/H| + 1) \odot \phi_H(R)$ is a K/H-coset, we conclude that

$$(0)(1) \cdot \ldots \cdot (|K/H| + 2) \odot T'Rg$$

contains the full K-coset

$$\left(\frac{(|K/H| + 1)(|K/H| + 2)}{2} - 1 \right) \alpha + (|K/H| + 2)g + K. \tag{3.22}$$

Likewise, since $(1)(2) \odot T' = H$ and $(3)(4) \cdot \ldots (|K/H| + 2) \odot \phi_H(R)$ is a K/H-coset, we conclude that

$$(0)(1) \cdot \ldots \cdot (|K/H| + 2) \odot T'Rg$$

also contains the full K-coset

$$\left(\frac{(|K/H| + 2)(|K/H| + 3)}{2} - 3 \right) \alpha + K. \tag{3.23}$$

As all terms of ST'^{-1} are from $\alpha + K$, we have $\phi_K(\alpha) = \phi_K(g)$, while in view of $|W \odot S| < 2|K|$, both K-cosets given in (3.22) and (3.23) must be equal; which implies $2\phi_K(\alpha) = 0$. As a result, we derive from (3.11) and $K < G$ that $|G/K| = 2$.

If $|K/H| \leq 2$, then $|G| = |G/K||K/H| \leq 2 \cdot 2 = 4$, contrary to assumption. Therefore we now conclude that $|K/H| \geq 3$. Next observe that

$$(0)(2) \odot T' + (1)(3)(4) \cdot \ldots \cdot (|K/H| + 2) \odot Rg \subseteq \left(\frac{(|K/H| + 2)(|K/H| + 3)}{2} - 2 \right) \alpha + K,$$

which is a K-coset disjoint from that of (3.23). Consequently,

$$|W \odot S| \geq |K| + |(0)(2) \odot T' + (1)(3)(4) \cdot \ldots \cdot (|K/H| + 2) \odot Rg|. \tag{3.24}$$

However, $(3)(4) \cdot \ldots \cdot (|K/H| + 2) \odot \phi_H(R)$ is a full K/H-coset (as previously derived by use of the induction hypothesis to define R), which readily implies that

$$|(0)(2) \odot T' + (1)(3)(4) \cdot \ldots \cdot (|K/H| + 2) \odot Rg| \geq |K/H| \geq 3.$$

Combined with (3.24) and (3.20), we conclude that $|W \odot S| \geq |K| + 3 \geq |S| + 1$, as desired. This completes the final subcase of Case 3. For the remainder of the arguments, we return to considering S translated so that $\mathsf{v}_0(S) = \mathsf{h}(S)$.

Case 4: $\frac{1}{3}(|S| + 2) \leq \mathsf{h}(S) \leq |S| - 3$.

Note that the case hypothesis implies $|S| \geq 6$. If $g \in \mathrm{supp}(S)$ is nonzero with $d := \mathrm{ord}(g) \leq \lceil \frac{1}{3}(|S|+2) \rceil$, then $0^d g \in \mathcal{F}(G)$ is a subsequence of S with length $|0^d g| = d + 1 = |\langle g \rangle| + 1 \leq |S| \leq |G|$; moreover, $\langle \mathrm{supp}(0^d g) \rangle_*$ is equal to the proper (since the previous inequality implies $d < |G|$), nontrivial subgroup $\langle g \rangle$. Consequently, Case 3 can be invoked to complete the proof. Therefore we instead conclude that

$$\mathrm{ord}(g) \geq \lceil \frac{1}{3}(|S| + 2) \rceil + 1 \quad \text{ for all nonzero } \ g \in \mathrm{supp}(S). \tag{3.25}$$

Since $\mathsf{v}_0(S) \leq |S| - 3$, choose some nonzero $x \in \mathrm{supp}(S)$. In view of Case 1, we have $|\mathrm{supp}(S)| \geq 3$, whence there must be some other nonzero $y \in \mathrm{supp}(S)$ with $x \neq y$. If, for every such nonzero $y \in \mathrm{supp}(S)$ with $x \neq y$, we have $y \in \langle x \rangle$, then $\langle x \rangle = \langle \mathrm{supp}(S) \rangle = G$. Otherwise, we can find some nonzero $y \in \mathrm{supp}(S)$ with $x \neq y$ and $\langle x, y \rangle > \langle x \rangle$. As a result, choosing the nonzero $y \in \mathrm{supp}(S) \setminus \{0, x\}$ appropriately and setting $K_1 = \langle x, y \rangle$, we obtain

$$|K_1| = |\langle x, y \rangle| \geq \min\{|G|,\, 2\,\mathrm{ord}(x)\} \geq \min\{|G|,\, 2\lceil \frac{1}{3}(|S| + 2) \rceil + 2\}, \tag{3.26}$$

where the latter bound follows from (3.25).

Let $R_1 = 0^{\lceil \frac{1}{3}(|S|+2) \rceil - 1} xy$. In view of the case hypothesis, we see that $R_1 \mid S$ with $0 \in \mathrm{supp}(SR_1^{-1})$. Let $R_2 = SR_1^{-1}$, so that, in view of $|S| \geq 6$ and the previous observation, we have

$$0 \in \mathrm{supp}(R_1) \cap \mathrm{supp}(R_2). \tag{3.27}$$

Let $K_2 = \langle \mathrm{supp}(R_2) \rangle_* = \langle \mathrm{supp}(R_2) \rangle$. In view of (3.27), we also have $K_1 = \langle \mathrm{supp}(R_1) \rangle_* = \langle \mathrm{supp}(R_1) \rangle$. Observe that

$$K_1 + K_2 = \langle \mathrm{supp}(S) \rangle = G. \tag{3.28}$$

From the case hypothesis $\mathsf{v}_0(S) \leq |S| - 3$ and (3.27), we see that $|\mathrm{supp}(R_2)| \geq 2$, whence K_2 is nontrivial. If $|K_2| \leq |R_2| - 1 \leq |S| - 1 \leq |G| - 1$, then K_2 will be proper and R_2 will be a sequence of length at least $|K_2| + 1$ all of whose terms come from the coset $0 + K_2$, whence Case 3 completes the proof. Therefore we can assume

$$|K_2| \geq |R_2| = |S| - |R_1| = |S| - \lceil \frac{1}{3}(|S| + 2) \rceil - 1 = \lfloor \frac{2|S| - 5}{3} \rfloor. \tag{3.29}$$

From (3.25), we also have

$$|K_2| \geq \lceil \frac{1}{3}(|S| + 2) \rceil + 1. \tag{3.30}$$

Let $A_1 = (0)(1) \cdot \ldots \cdot (|R_1| - 1) \odot R_1$ and let $A_2 = (|R_1|)(|R_1| + 1) \cdot \ldots \cdot (|S| - 1) \odot R_2$. In view of Lemma 3.3, we have $\langle A_1 \rangle_* = \langle \mathrm{supp}(R_1) \rangle_* = K_1$ and $\langle A_2 \rangle_* = \langle \mathrm{supp}(R_2) \rangle_* = K_2$. Also, from their definition, we have

$$A_1 + A_2 \subseteq W \odot S. \tag{3.31}$$

Since $|\mathrm{supp}(R_2)| \geq 2$, it is readily deduced that $|A_2| \geq 2$. In consequence, if $|A_1| \geq |G| - 1$, then applying Lemma 2.2 to $A_1 + A_2$ shows that $A_1 + A_2 = G$, which in view of (3.31) completes the proof. Therefore we can assume $|A_1| \leq |G| - 2$. Consequently, in view of $\langle \mathrm{supp}(R_1) \rangle_* = K_1$ and (3.26), applying Lemma 3.1(iii) to R_1 results in

$$|A_1| \geq 2|R_1| - 2 = 2\lceil \frac{1}{3}(|S| + 2) \rceil. \tag{3.32}$$

Since $|R_2| < |S|$, we can apply the induction hypothesis to R_2 to yield

$$|A_2| \geq \min\{|K_2| - 1,\, |R_2|\} = \min\{|K_2| - 1,\, \lfloor \frac{2|S| - 5}{3} \rfloor\} \geq \lfloor \frac{2|S| - 5}{3} \rfloor - 1, \tag{3.33}$$

where the final inequality follows from (3.29).

If $|A_1 + A_2| \geq |A_1| + |A_2| - 1$, then (3.32), (3.33) and (3.31) together yield

$$|W \odot S| \geq 2|R_1| - 2 + |R_2| - 2 = |S| + |R_1| - 4 = |S| + \lceil \frac{1}{3}(|S| + 2) \rceil - 3,$$

which is at least $|S|$ for $|S| \geq 6$, as desired. So we can instead assume

$$|A_1 + A_2| < |A_1| + |A_2| - 1. \tag{3.34}$$

Let $H = \mathsf{H}(A_1 + A_2)$ be the maximal period of $A_1 + A_2$. In view of (3.34) and Kneser's Theorem, it follows that H is a proper (else $W \odot S = G$, as desired), nontrivial subgroup with

$$|\phi_H(A_1 + A_2)| \geq |\phi_H(A_1)| + |\phi_H(A_2)| - 1. \tag{3.35}$$

We divide the remainder of the case into several subcases.

Subcase 4.1: $|\phi_H(A_1)| = |\phi_H(A_2)| = 1$.
In this case, $K_1 = \langle A_1 \rangle_* \leq H$ and $K_2 = \langle A_2 \rangle_* \leq H$, whence $G = K_1 + K_2 \leq H$ follows from (3.28), contradicting that $H < G$ is proper.

Subcase 4.2: $|\phi_H(A_1)| \geq 2$ and $|\phi_H(A_2)| = 1$.
In this case, $K_2 = \langle A_2 \rangle_* \leq H$ and $|A_1 + A_2| \geq 2|H| \geq 2|K_2|$, which is at least $\frac{4}{3}|S| - \frac{14}{3}$ in view of (3.29). For $|S| \geq 12$, combing this with (3.31) implies $|W \odot S| \geq |A_1 + A_2| > |S| - 1$, as desired. For $|S| \leq 11$, we can use (3.30) and (3.31) to estimate $|W \odot S| \geq |A_1 + A_2| \geq 2|K_2| \geq \frac{2}{3}|S| + \frac{10}{3} > |S| - 1$, also as desired.

Subcase 4.3: $|\phi_H(A_2)| \geq 2$.
In this case, (3.35) and (3.31) imply

$$|W \odot S| \geq |A_1 + A_2| \geq |A_1 + H| + |A_2 + H| - |H| \geq |A_1 + H| + \frac{1}{2}|A_2 + H| \geq |A_1| + \frac{1}{2}|A_2|.$$

Combined with (3.32) and (3.33), we obtain

$$|W \odot S| \geq 2\lceil \frac{1}{3}(|S| + 2) \rceil + \frac{1}{2}(\lfloor \frac{2|S| - 5}{3} \rfloor - 1) > |S| - 1,$$

as desired, which completes the last subcase of Case 4.

Case 5: $\mathsf{h}(S) \leq \frac{1}{3}(|S| + 1)$.
Let

$$\epsilon = \begin{cases} 1, & \text{if } |S| \equiv 2 \mod 3 \\ 0, & \text{else} \end{cases}$$

and let $r = \lfloor \frac{1}{3}(|S| + 1) \rfloor$. Note $r \geq 1$ in view of $|S| \geq 4$. We assume by contradiction that S fails to satisfy the theorem (solely for the statements of the properties below, which might not hold if S satisfied the conditions of the theorem).

The assumption $\mathsf{h}(S) \leq \frac{1}{3}(|S| + 1)$ allows us to factorize the sequence S into square-free subsequences in the following way (this is the basic construction for the existence of an r-setpartition; see [11]):

- If $|S| \equiv 0 \mod 3$, then $r = \frac{1}{3}|S|$, $\epsilon = 0$, and we can factorize $S = S_1 \cdot \ldots \cdot S_r$ such that $|\operatorname{supp}(S_i)| = |S_i| = 3$ for all $i \in [1, r]$.
- If $|S| \equiv 1 \mod 3$, then $r = \frac{1}{3}(|S| - 1)$, $\epsilon = 0$, and we can factorize $S = S_1 \cdot \ldots \cdot S_r S_{r+1}$ such that $|\operatorname{supp}(S_i)| = |S_i| = 3$ for all $i \in [1, r]$ and $|\operatorname{supp}(S_{r+1})| = |S_{r+1}| = 1$.
- If $|S| \equiv 2 \mod 3$, then $r = \frac{1}{3}(|S| + 1)$, $\epsilon = 1$, and we can factorize $S = S_1 \cdot \ldots \cdot S_r$ such that $|\operatorname{supp}(S_i)| = |S_i| = 3$ for all $i \in [1, r - 1]$ and $|\operatorname{supp}(S_r)| = |S_r| = 2$.

Note ϵ counts the number of S_i with length 2 in the factorization. For the purposes of the proof, we will refer to a factorization $S_1 \cdot \ldots \cdot S_r$ (of S or SS_{r+1}^{-1}) as *well-balanced* if it satisfies the above criteria and also has $|\langle \operatorname{supp}(S_j) \rangle_*| \geq 5$ for any S_j with $|S_j| \geq 3$. Let us show that such a factorization exists.

Let $S_1 \cdot \ldots \cdot S_r \mid S$ be a factorization satisfying the appropriate bulleted criteria above. We trivially have $|\langle \operatorname{supp}(S_j) \rangle_*| \geq 3$ for each S_j with $|S_j| = |\operatorname{supp}(S_j)| = 3$. If $|\langle \operatorname{supp}(S_j) \rangle_*| = 4$, then the pigeonhole principle guarantees that there are distinct $x, y \in \operatorname{supp}(S_j)$ with $\operatorname{ord}(x - y) = 2$, whence invoking Case 3 with $H = \langle x - y \rangle$ shows that the theorem holds for S, contrary to assumption. Therefore, we see that $|\langle \operatorname{supp}(S_j) \rangle_*| \geq 5$ or $|\langle \operatorname{supp}(S_j) \rangle_*| = 3$ for each S_j with $|S_j| = 3$. Consider a factorization $S_1 \cdot \ldots \cdot S_r \mid S$ satisfying the appropriate bulleted criteria so that the number of S_j with $|S_j| = |\langle \operatorname{supp}(S_j) \rangle_*| = 3$ is minimal. If by contradiction no well-balanced factorization exists, then there will be some S_j with $|S_j| = |\langle \operatorname{supp}(S_j) \rangle_*| = 3$. Thus $\operatorname{supp}(S_j)$ is a coset of the cardinality 3 subgroup $H := \langle \operatorname{supp}(S_j) \rangle_*$. In view of $|S| \geq 4$, there is some S_k with $k \in [1, r + 1]$, $k \neq j$, and $k = r + 1$ only if $|S| = 4 \equiv 1 \mod 3$. If $\operatorname{supp}(S_k)$ and $\operatorname{supp}(S_j)$ share a common element, then there will be 4 terms of S from the same cardinality three H-coset, whence invoking Case 3 shows that the theorem holds for S, contrary to assumption.

Therefore we may instead assume that $\mathrm{supp}(S_k)$ and $\mathrm{supp}(S_j)$ are disjoint. Thus if we swap any term x from S_j for a term y from S_k and let $S'_j = S_j x^{-1} y$ and $S'_k = S_k y^{-1} x$ denote the resulting sequences, then Lemma 3.4 guarantees that $\mathrm{supp}(S'_j)$ cannot be periodic. In particular, $\mathrm{supp}(S'_j)$ is not a coset of a cardinality 3 subgroup. If $\mathrm{supp}(S'_k)$ is also not a coset of a cardinality three subgroup, then set $S''_j = S'_j$ and $S''_k = S'_k$. On the other hand, if $\mathrm{supp}(S'_k)$ is a coset of a cardinality 3 subgroup, then Lemma 3.4 again shows that $S''_k := S'_k {y'}^{-1} y$ is not periodic, and thus not coset of cardinality 3 subgroup, where y' is any element from $\mathrm{supp}(S_k)$ distinct from y. Moreover, we also have $S''_j := S'_j y^{-1} y' = S_j x^{-1} y'$ not being a coset of a cardinality three subgroup (by repeating the arguments used to show this for S'_j only using y' instead of y). However, now the factorization $S_1 \cdot \ldots \cdot S_r S_j^{-1} S_k^{-1} S''_j S''_k$ satisfies the appropriate bulleted condition and also has at least one less S_j with $|S_j| = |\langle \mathrm{supp}(S_j)\rangle_*| = 3$, contradicting the assumed minimality assumption. This shows that a well-balanced factorization $S_1 \cdot \ldots \cdot S_r$ exists.

For the moment, let $S_1 \cdot \ldots \cdot S_r \mid S$ be an arbitrary well-balanced factorization. Let $W = W_1 \cdot \ldots \cdot W_r$ be a factorization of W with $|W_i| = |S_i|$ for all $i \in [1, r]$ such that each W_i is a sequence of consecutive integers. Note we can apply Lemma 3.2 to each S_j with $|S_j| = 3$ since the definition of a well-balanced factorization ensures that $|\langle \mathrm{supp}(S_j)\rangle_*| \ge 5$ while we have $\mathrm{ord}(x - y) \ge 3$ for all distinct $x, y \in \mathrm{supp}(S_j)$, else Case 3 applied with $H = \langle x - y\rangle$ shows that the theorem holds for S, contrary to assumption. For each S_j with $|S_j| = 3$, let $A_j \subseteq W_j \odot S_j$ be the resulting subset with

$$|A_j| = 4, \quad \langle A_j\rangle_* = \langle \mathrm{supp}(S_j)\rangle_*, \quad \text{and either} \quad \langle A_j\rangle_* \cong C_6 \quad \text{or} \quad |\mathsf{H}(A_j)| \neq 2. \tag{3.36}$$

For any S_j with $|S_j| \neq 3$, let $A_j = W_j \odot S_j$. If $|S| \not\equiv 1 \mod 3$, set $A_{r+1} = \{0\}$. Note that $|A_{r+1}| = 1$ (regardless of the value of $|S|$ modulo 3) and that $|A_r| = 2$ when $|S_r| = 2$. We also have

$$\sum_{i=1}^{r+1} A_i \subseteq W \odot S. \tag{3.37}$$

For the purposes of the proof, we will refer to a setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_r A_{r+1}$ obtained as above from a well-balanced factorization $S_1 \cdot \ldots \cdot S_r \mid S$ as a *well-balanced* setpartition.

Our plan is to show that a well-balanced setpartition with maximal cardinality sumset has $|\sum\limits_{i=1}^{r+1} A_i| \ge |S|$, which in view of (3.37) will yield the concluding contradiction $|W \odot S| \ge |S|$. To do this, we must first establish some properties that any well balanced setpartition has. We begin with the following.

Property 1: If $A_1 \cdot \ldots \cdot A_r A_{r+1}$ is a well-balanced setpartition and

$$|\sum_{i\in I} A_i| < \sum_{i \in I} |A_i| - |I| + 1, \tag{3.38}$$

where $I \subseteq [1, r]$ is a nonempty subset, then $|\mathsf{H}(\sum\limits_{i\in I} A_i)| \ge 5$.

Let $H = \mathsf{H}(\sum\limits_{i\in I} A_i)$ and suppose by contradiction that $|H| \le 4$. In view of (3.38) and Kneser's Theorem, we know $|H| \ge 2$ with

$$|\sum_{i\in I} A_i| \ge \sum_{i\in I} |A_i| - |I| + 1 - (|I| - 1)(|H| - 1) + \rho,$$

where $\rho = \sum\limits_{i\in I}(|A_i + H| - |A_i|)$ denotes the number of H-holes in the A_i with $i \in I$. In particular,

$$\rho < (|I| - 1)(|H| - 1). \tag{3.39}$$

Suppose $|H| \in \{3, 4\}$. Now all but at most one A_i with $i \in I \subseteq [1, r]$ has $|A_i| = 4$. Since $|\langle A_i\rangle_*| = |\langle \mathrm{supp}(S_i)\rangle_*| \ge 5 > |H|$ for such A_i, we know that each such A_i intersects at least two H-cosets, whence

$$|A_i + H| - |A_i| \ge 2|H| - 4 \ge |H| - 1.$$

Thus $\rho = \sum\limits_{i\in I}(|A_i + H| - |A_i|) \ge (|I| - 1)(|H| - 1)$, contradicting (3.39). So we may instead assume $|H| = 2$.

If A_i is H-periodic with $|A_i| = 2$, then Lemma 3.3 implies that S_i consists of 2 distinct elements from the same cardinality 2 H-coset, whence applying Case 3 shows that the theorem holds for S, contrary to assumption. Therefore only A_i with $|A_i| = 4$ can be H-periodic.

If at most one A_i with $i \in I$ is H-periodic, then $|A_i + H| - |A_i| \ge 1 = |H| - 1$ will hold for all but at most one $i \in I$, and we will again contradict (3.39). Therefore there must be at least two A_i with $i \in I$ that are H-periodic, and in view of the previous paragraph, we must have $|A_i| = 4$ for each such A_i. However (3.36) shows this is only possible for A_i if $\langle \mathrm{supp}(S_i)\rangle_* = \langle A_i\rangle_* \cong C_6$, in which case A_i is a cardinality 4 subset of a coset of the cardinality 6 subgroup $\langle A_i\rangle_*$.

Let $J \subseteq I$ be the subset of all those indices $i \in I$ such that A_i is H-periodic. Since there are at least two A_i with $i \in I$ and A_i being H-periodic, as shown above, we have $|J| \geq 2$. By the argument of the previous paragraph, each A_i with $i \in J$ has $\langle \phi_H(A_i)\rangle_* \cong C_3$. Thus, if $\langle \phi_H(A_i)\rangle_* = \langle \phi_H(A_j)\rangle_*$ for distinct $i, j \in J$, then Lemma 2.2 implies that $A_i + A_j$ is $\langle A_j\rangle_*$-periodic, contradicting that $H < \langle A_j\rangle_*$ is the maximal period of $\sum\limits_{i\in I} A_i$. Therefore we may assume each $\langle \phi_H(A_i)\rangle_*$, for $i \in J$, is a distinct cardinality 3 subgroup. In consequence, we have

$$|\phi_H(A_i) + \phi_H(A_j)| = 4 \quad \text{for } i, j \in J \text{ distinct.} \tag{3.40}$$

Since H is the maximal period of $\sum\limits_{i\in I} A_i$ and $J \subseteq I$, it follows that $\sum\limits_{i\in J} \phi_H(A_i)$ is aperiodic. Thus, pairing up the $\phi_H(A_j)$ with $j \in J$ into $\lfloor \frac{1}{2}|J|\rfloor$ pairs, applying the equality (3.40) to each pair, and then applying Kneser's Theorem to the aperiodic $\lceil \frac{1}{2}|J|\rceil$-term sumset whose summands consist of the sumsets of each of the $\lfloor \frac{1}{2}|J|\rfloor$ pairs along with the one unpaired set $\phi_H(A_i)$ with $i \in J$ (if $|J|$ is odd) yields the estimates

$$|\sum_{i\in J} \phi_H(A_i)| \geq 4\left(\frac{|J|-1}{2}\right) + 2 - \frac{|J|+1}{2} + 1 = \frac{3}{2}|J| + \frac{1}{2} \qquad \text{if } |J| \text{ is odd,} \tag{3.41}$$

$$|\sum_{i\in J} \phi_H(A_i)| \geq 4\left(\frac{|J|}{2}\right) - \frac{1}{2}|J| + 1 = \frac{3}{2}|J| + 1 \qquad \text{if } |J| \text{ is even.}$$

For each $i \in I \setminus J \subseteq [1, r]$, we know A_i is not H-periodic. As a result, if $i \in I \setminus J$ with $|A_i| = 4$, then $|\phi_H(A_i)| \geq 3$, while if $i \in I \setminus J$ with $|A_i| = 2$, then $|\phi_H(A_i)| = 2$. Consequently, since $\sum\limits_{i\in I} \phi_H(A_i)$ is aperiodic (as H is the maximal period of $\sum\limits_{i\in I} A_i$), Kneser's Theorem and (3.41) together imply

$$|\sum_{i\in I} \phi_H(A_i)| \geq |\sum_{i\in J} \phi_H(A_i)| + \sum_{i\in I\setminus J} |\phi_H(A_i)| - (|I\setminus J|+1) + 1 \geq \frac{3}{2}|J| + \frac{1}{2} + 2|I\setminus J| - \epsilon \geq \frac{3}{2}|I| + \frac{1}{2} - \epsilon. \tag{3.42}$$

Since $\sum\limits_{i\in I} A_i$ is H-periodic with $|H| = 2$, (3.42) implies $|\sum\limits_{i\in I} A_i| \geq 3|I| + 1 - 2\epsilon = \sum\limits_{i\in I} |A_i| - |I| + 1$, contradicting (3.38) and completing the proof of Property 1.

Next, recalling the definition of r, we observe that

$$\sum_{i=1}^{r} |A_i| - r + 1 = 4r - 2\epsilon - r + 1 = 3r - 2\epsilon + 1 \geq |S|.$$

Consequently, in view of (3.37) and $W \odot S \neq G$, it follows that

$$|\sum_{i=1}^{r} A_i| < \min\{|G|, \sum_{i=1}^{r} |A_i| - r + 1\}. \tag{3.43}$$

Thus Property 1 ensures that $H_1 := \mathsf{H}(\sum\limits_{i=1}^{r} A_i)$ has $|H_1| \geq 5$. Since H_1 must be a proper subgroup, it follows that $|G|$ is composite with

$$|G| \geq 2|H_1| \geq 10.$$

Let $I_1 \subseteq [1, r]$ denote all those indices $i \in [1, r]$ such that $|\phi_{H_1}(A_i)| = 1$. Our next goal is the following.

Property 2: If $A_1 \cdot \ldots \cdot A_r A_{r+1}$ is a well-balanced setpartition with $H_1 = \mathsf{H}(\sum\limits_{i=1}^{r} A_i)$ and $I_1 \subseteq [1, r]$ being the subset of all $i \in [1, r]$ with $|\phi_{H_1}(A_i)| = 1$, then $|I_1| \geq \lceil \frac{1}{3}(|H_1| - 2)\rceil + 2$.

First let us handle the case when $|I_1| = r = \lfloor \frac{|S|+1}{3} \rfloor$. In this case, we need to show $|S| \geq |H_1| + 5$, for which, in view of $|W \odot S| < |S|$, it suffices to show that $|W \odot S| \geq |H_1| + 4$. Since $|\sum\limits_{i=1}^{r} A_i| \leq |W \odot S| < |S|$, we have the initial estimate $|S| \geq |H_1| + 1$. However, if $|W \odot S| = |H_1|$, then $\langle \mathrm{supp}(S)\rangle_* = \langle W \odot S\rangle_* = H_1 < G$ follows from Lemma 3.3, contradicting the hypothesis $\langle \mathrm{supp}(S)\rangle_* = G$. Therefore we instead conclude that $|W \odot S| \geq |H_1| + 1$, in turn implying

$$|S| \geq |W \odot S| + 1 \geq |H_1| + 2 \geq 7. \tag{3.44}$$

Since $|I_1| = r$, we know that every A_i with $i \in [1, r]$ is contained in an H_1-coset. Consequently, in view of (3.36), we see that each S_i with $i \in [1, r]$ has all its terms from a single H_1-coset, say $\mathrm{supp}(S_i) \subseteq \alpha_i + H_1$. If it is the same H_1-coset for all S_i with $i \in [1, r]$, then we will have at least $|S| - 1 \geq |H_1| + 1$ terms from the same H_1-coset (in view of (3.44)), whence Case 3 shows that the theorem holds for S, contrary to assumption. Therefore we can instead assume $\alpha_j + H_1 \neq \alpha_r + H_1$ for some $j \in [1, r-1]$. Let $g_r \in \mathrm{supp}(S_r)$

and $g_j \in \text{supp}(S_j)$ and define $A'_r = W_r \odot S_r g_r^{-1} g_j$ and $A'_j = W_j \odot S_j g_j^{-1} g_r$. For $i \in [1, r+1] \setminus \{r, j\}$, set $A'_i = A_i$. Then, since neither $S_r g_r^{-1} g_j$ nor $S_j g_j^{-1} g_r$ is contained in a single H_1-coset, it follows from Lemma 3.3 that $|\phi_{H_1}(A'_j)| \geq 2$ and $|\phi_{H_1}(A'_r)| \geq 2$. In consequence, the subset $\sum_{i=1}^{r+1} A'_i \subseteq W \odot S$ intersects at least two H_1-cosets, one of which must be disjoint from the H_1-coset that contained $\sum_{i=1}^{r+1} A_i$.

If $r \geq 3$, then there will be some $A_i = A'_i$ with $i \in [1, r-1] \setminus \{j\}$, which will be a cardinality 4 subset of a single H_1-coset, thus ensuring that every H_1-coset that intersects $\sum_{i=1}^{r+1} A'_i$ must contain at least 4 elements. As a result, if $r \geq 3$, then $|W \odot S| \geq |H_1| + 4$, as desired. Therefore it remains to consider the case when $r \leq 2$ in order to finish the case when $|I_1| = r$. However, (3.44) shows that $r \leq 2$ is only possible if $|H_1| = 5$, $|S| = 7$, $r = 2$ and $j = 1$. In this case, $|S| \equiv 1 \mod 3$, so that S_{r+1} contains a term from S. Since $\alpha_1 + H_1 = \alpha_j + H_1 \neq \alpha_r + H_1 = \alpha_2 + H_1$, we can w.l.o.g. assume $\alpha_2 + H_1 \neq \alpha_3 + H_1$, where α_3 is the single term from S_3. But now, defining $A''_1 = A_1 \subseteq (0)(1)(2) \odot S_1$, $A''_2 = (3)(4) \odot S_2 g_2^{-1}$ and $A''_3 = (5)(6) \odot S_r g_2$, we can repeat the arguments from the $r \geq 3$ case using the A''_i instead of the A'_i in order to conclude $|W \odot S| \geq |H_1| + 4$ in this final remaining case as well. So, for the remainder of the proof of Property 2, we can now assume $|I_1| \leq r - 1$.

From Kneser's Theorem, (3.37), the definitions of I_1 and r, and the assumption $|W \odot S| < |S|$, we have

$$|S| - 1 \geq |\sum_{i=1}^{r} A_i| \geq (r - |I_1| + 1)|H_1| = (\lfloor \frac{|S|+1}{3} \rfloor - |I_1| + 1)|H_1|, \tag{3.45}$$

from which we derive both

$$|I_1| \geq \lfloor \frac{|S|+1}{3} \rfloor + 1 - \frac{|S|-1}{|H_1|} \geq (|S| - 1)\frac{|H_1| - 3}{3|H_1|} + 1$$

and $|S| \geq (e+1)|H_1| + 1$, where $e := r - |I_1| \geq 1$. Combining these inequalities yields

$$|I_1| \geq (e+1)\frac{|H_1|}{3} - e.$$

Since $|H_1| \geq 5$, the above bound is minimized for small e. Thus, since $e \geq 1$, we obtain

$$|I_1| \geq \lceil \frac{2}{3}|H_1| \rceil - 1, \tag{3.46}$$

which is at least the desired bound $\lceil \frac{1}{3}(|H_1| - 2) \rceil + 2$ except when $|H_1| = 6$. In this case, we must have $|S| = 2|H_1| + 1 = 13$ with $e = 1$, else the estimate (3.46) will become strict, yielding the desired bound on $|I_1|$. Thus $r = 4$.

Since $|S| = 13 \equiv 1 \mod 3$, the set S_{r+1} contains a term from S, say α_{r+1}. In view of (3.36) and the definition of I_1, we know each $\text{supp}(S_i)$, for $i \in I_1$, is contained in a single H_1-coset. If this single H_1-coset is equal to $\alpha_{r+1} + H$ for each $i \in I_1$, then we will have $3|I_1| + 1 = 10 \geq |H_1| + 1$ terms of S from the same H_1-coset, whence invoking Case 3 shows that the theorem holds for S, contrary to assumption. Therefore there must be some $j \in I_1$ such that $\text{supp}(S_j) \subseteq \alpha_j + H_1 \neq \alpha_{r+1} + H_1$, say w.l.o.g. $j = r$. Set $A'_i = A_i$ for $i \in [1, r-1]$, set $A'_r = (9)(10) \odot S_r g^{-1}$ and set $A'_{r+1} = (11)(12) \odot S_{r+1} g$, where $g \in \text{supp}(S_r)$. Observe that $\phi_{H_1}(A_i) = \phi_{H_1}(A'_i)$ for $i \in [1, r-1]$ while $|\phi_{H_1}(A_r)| = |\phi_{H_1}(A'_r)| = 1$. Consequently, $\sum_{i=1}^{r} \phi_H(A'_i)$ is a translate of $\sum_{i=1}^{r} \phi_H(A_i)$; in particular, $\sum_{i=1}^{r} \phi_{H_1}(A'_i)$ is aperiodic in view of H_1 being the maximal period of $\sum_{i=1}^{r} A_i$. However, since $\text{supp}(S_{r+1} g)$ is not contained in a single H_1-coset, it follows from Lemma 3.3 that $|\phi_{H_1}(A'_{r+1})| \geq 2$, whence, since $\sum_{i=1}^{r} \phi_{H_1}(A'_i)$ is aperiodic, Kneser's Theorem implies that

$$|\sum_{i=1}^{r+1} \phi_{H_1}(A'_i)| > |\sum_{i=1}^{r} \phi_{H_1}(A'_i)| = |\sum_{i=1}^{r} \phi_{H_1}(A_i)| = |\sum_{i=1}^{r+1} \phi_{H_1}(A_i)|.$$

Thus $\sum_{i=1}^{r+1} A'_i \subseteq W \odot S$ intersects some H_1-coset that is disjoint from $\sum_{i=1}^{r+1} A_i \subseteq W \odot S$, which combined with (3.45) and the definition of e implies that

$$|S| > |W \odot S| > |\sum_{i=1}^{r+1} A_i| = |\sum_{i=1}^{r} A_i| \geq (e+1)|H_1| = 12,$$

yielding the contradiction $|S| \geq 14$. Thus Property 2 is established in the final remaining case.

Property 3: Let $A_1 \cdot \ldots \cdot A_r A_{r+1}$ be a well-balanced setpartition, let $K \leq G$ be a subgroup, let $J \subseteq [1, r]$ be a subset of indices with $|\phi_K(A_i)| = 1$ and $|A_i| = 4$ for all $i \in J$, let $L = \mathsf{H}(\sum\limits_{i \in J} A_i)$, and let $I \subseteq J$ denote all those indices $i \in J$ with $|\phi_L(A_i)| = 1$. If $|J| \geq \lceil \frac{1}{3}(|K| - 2) \rceil$ and $5 \leq |L| < |K|$, then $|I| \geq \lceil \frac{1}{3}(|L| - 2) \rceil + 2$.

Since $|\phi_K(A_i)| = 1$ for all $i \in J$, each A_i with $i \in J$ is contained in a single K-coset, whence $\sum\limits_{i \in J} A_i$ is also contained in a single K-coset. Thus $L \leq K$, so that our hypothesis $|L| < |K|$ implies $|L| \leq \frac{1}{2}|K|$. In particular,

$$|K| \geq 2|L| \geq 10.$$

Suppose by contradiction that $|I| \leq \lceil \frac{1}{3}(|L| - 2) \rceil + 1 \leq \frac{1}{3}|L| + 1$. For each $i \in J \setminus I$, we have $|\phi_L(A_i)| \geq 2$. Thus, in view of $L \neq K$, Kneser's Theorem implies that $|J \setminus I| = |J| - |I| \leq |K/L| - 2$. Combined with our assumption on the size of $|I|$ and the hypothesis for the size of $|J|$, we find that

$$\left\lceil \frac{|K| - 2}{3} \right\rceil - |K/L| + 2 \leq |I| \leq \left\lceil \frac{|L| - 2}{3} \right\rceil + 1, \tag{3.47}$$

which implies $\frac{1}{3}|K| \leq \frac{1}{3}|L| + |K/L| - \frac{1}{3}$, in turn yielding

$$|K| \leq |L| + \frac{3|K|}{|L|} - 1. \tag{3.48}$$

Considering the right hand side of (3.48) as a function of $|L|$, we find that its maximum will be obtained for a boundary value of $|L|$, i.e., for $|L| = 5$ or $|L| = \frac{1}{2}|K|$. If $|L| = \frac{1}{2}|K|$, we obtain $|K| \leq \frac{1}{2}|K| + 5$, and if $|L| = 5$, we obtain $|K| \leq \frac{3}{5}|K| + 4$. In view of $|K| \geq 10$, both of these inequalities can only hold for $|K| = 10$ with $|L| = 5$ (in view of $|L| \geq 5$). However, for these values, we see that (3.47) instead implies $3 - 2 + 2 \leq 2$, a contradiction. Thus Property 3 is established.

With the above three properties established for an arbitrary well-balanced setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_r A_{r+1}$, we now proceed to complete the proof by considering a well-balanced setpartition satisfying an iterated list of extremal conditions. The argument that follows is a simple variation of the basic strategy used to proof the Partition Theorem [24]. During the course of the construction of $\mathcal{A}$, we will at times declare certain quantities fixed, by which we mean that any additional assumption on $\mathcal{A}$ is always subject to all previously fixed quantities being maintained in their current state.

We begin by setting $J_1 = [1, r]$, fixing S_{r+1}, and assuming our well-balanced setpartition $A_1 \cdot \ldots \cdot A_r A_{r+1}$ has maximal cardinality sumset $|\sum\limits_{i \in J_1} A_i| < |S| \leq |G|$ (in view of $|W \odot S| < |S|$). Fix $\sum\limits_{i \in J_1} A_i$ up to translation. Let $H_1 = \mathsf{H}(\sum\limits_{i \in J_1} A_i)$ and I_1 be as defined above Property 2.

Next assume that $|I_1|$ is minimal (subject to all prior fixed quantities and extremal assumptions). We showed above that $H_1 = \mathsf{H}(\sum\limits_{i=1}^{r} A_i)$ has $|H_1| \geq 5$, while Property 2 ensures that $|I_1| \geq \lceil \frac{1}{3}(|H_1| - 2) \rceil + 2$. We have $\langle A_i \rangle_* \subseteq H_1$ for all $i \in I_1$, whence (3.36) ensures that $\langle \mathrm{supp}(S_i) \rangle_* \subseteq H_1$ for all $i \in I_1$. Thus each $\mathrm{supp}(S_i)$, for $i \in I_1$, is contained in some H_1-coset. If it is the same H_1-coset for every $i \in I_1$, then we will have at least $3|I_1| - \epsilon \geq 3(\frac{1}{3}(|H_1| - 2) + 2) - \epsilon \geq |H_1| + 1$ terms of S all from the same H_1-coset, whence Case 3 applied using the group $\langle \mathrm{supp}(\prod_{i \in I_1} S_i) \rangle_* \leq H_1 < G$ shows that the theorem holds for S, contrary to assumption. Therefore we may instead assume that there are distinct $k_1, k_1' \in I_1$ with $\mathrm{supp}(S_{k_1})$ and $\mathrm{supp}(S_{k_1'})$ contained in distinct H_1-cosets; moreover, if $|A_j| = 2$ for some $j \in I_1$, then we can additionally assume $j \in \{k_1, k_1'\}$. Let $J_2 = I_1 \setminus \{k_1, k_1'\}$. Note $|A_i| = 4$ for all $i \in J_2$.

Fix S_i for all $i \in [1, r] \setminus J_2$, next assume that $|\sum\limits_{i \in J_2} A_i|$ is maximal subject to all prior extremal assumptions still holding, and then fix $\sum\limits_{i \in J_2} A_i$ up to translation. In view of $|J_2| = |I_1| - 2 \geq \lceil \frac{1}{3}(|H_1| - 2) \rceil$ and $|H_1| \geq 5$, we see that $|J_2|$ is nonempty. Moreover, we have

$$\sum_{i \in J_2} |A_i| - |J_2| + 1 = 3|J_2| + 1 \geq |H_1| - 1. \tag{3.49}$$

Let us next show that $|\sum\limits_{i \in J_2} A_i| < |H_1| - 1$. Suppose this is not the case: $|\sum\limits_{i \in J_2} A_i| \geq |H_1| - 1$. Now $\mathrm{supp}(S_{k_1})$ and $\mathrm{supp}(S_{k_1'})$ are contained in disjoint H_1-cosets. Consequently, if we can swap a term between S_{k_1} and $S_{k_1'}$ with the result giving a well-balanced setpartition satisfying all extremal assumptions coming before the assumption on $|\sum_{i\in J_2} A_i|$, then we will have contradicted the minimality of $|I_1|$. We proceed to do so.

Let $x \in \mathrm{supp}(S_{k_1})$ and let $y \in \mathrm{supp}(S_{k_1'})$. If swapping the terms x and y does not result in a well-balanced factorization, then w.l.o.g. we must have $|S_{k_1}| = 3$ with $\mathrm{supp}(S_{k_1}x^{-1}y)$ a coset of a cardinality 3 subgroup (as argued in the existence of a well-balanced setpartition). However, in view of Lemma 3.4, this means that $\mathrm{supp}(S_{k_1}x^{-1}y')$ is not periodic, and thus not a coset of cardinality 3 subgroup, for all other $y' \in \mathrm{supp}(S_{k_1'}y^{-1})$. Moreover, if $|\,\mathrm{supp}(S_{k_1'})| = 3$, then Lemma 3.4 also ensures that $S_{k_1'}xy'^{-1}$ cannot be a coset of a cardinality 3 subgroup for both remaining terms $y' \in \mathrm{supp}(S_{k_1'}y^{-1})$. Thus, for any $x \in \mathrm{supp}(S_{k_1})$, we can find a $y \in \mathrm{supp}(S_{k_1'})$ such that swapping x for y results in a well-balanced factorization, thus inducing a well-balanced setpartition where $A'_{k_1} \subseteq W_{k_1} \odot (A_{k_1}x^{-1}y)$ and $A'_{k_1'} \subseteq W_{k_1'} \odot (A_{k_1'}y^{-1}x)$ are obtained via Lemma 3.2 and have replaced A_{k_1} and $A_{k_1'}$. Furthermore, either $|A'_{k_1}| = 4$ or $|A'_{k_1'}| = 4$, say $|A'_{k_1}| = 4$, and then the construction of A'_{k_1} given by Lemma 3.2 allows us to assume there is a 2 element subset of A'_{k_1} contained in an H_1-coset.

Since $|\sum_{i\in J_2} A_i| \geq |H_1| - 1$, Lemma 2.2 implies that $\sum_{i\in I_1} A_i$ was a full H_1-coset (it cannot be larger as all sets A_i with $i \in J_2 \subseteq I_1$ are each themselves contained in an H_1-coset). However, since A'_{k_1} still contains two elements from an H_1-coset, Lemma 2.2 also ensures that $\sum_{i\in J_2} A_i + A_{k_1'} + A_{k_2'}$ contains a translate of this H_1-coset. Thus an appropriate translate of the sumset of the new setpartition contains all elements of $\sum_{i=1}^{r} A_i$, whence the maximality of $|\sum_{i=1}^{r} A_i|$ ensures that the sumset has not changed up to translation. Hence, since there are two less sets contained in a single H_1-coset in the new setpartition, we see that we have contradicted the minimality of $|I_1|$. So we instead conclude that $|\sum_{i\in J_2} A_i| < |H_1| - 1$, as claimed, which, in view of (3.49), implies that

$$|\sum_{i\in J_2} A_i| < \min\{|H_1|,\ \sum_{i\in J_2} |A_i| - |J_2| + 1\}. \tag{3.50}$$

In view of (3.50) and Property 1, we see that $H_2 := (\sum_{i\in J_2} A_i)$ has $5 \leq |H_2| < |H_1|$. Let $I_2 \subseteq J_2$ be all those indices $i \in J_2$ with $|\phi_{H_2}(A_i)| = 1$. Assume $|I_2|$ is minimal (subject to all prior fixed quantities and extremal assumptions). Since $|J_2| = |I_1| - 2 \geq \lceil \frac{1}{3}(|H_1| - 2)\rceil$, we can apply Property 3 (with $L = H_2$ and $K = H_1$) to conclude $|I_2| \geq \lceil \frac{1}{3}(|H_2| - 2)\rceil + 2$. As before, all terms A_i with $i \in I_2$ are contained in a single H_2-coset but not all in the same H_2-coset, else applying Case 3 shows that the theorem holds for S, contrary to assumption. This allows us to find $k_2,\ k_2' \in I_2$ such that A_{k_2} and $A_{k_2'}$ are contained in disjoint H_2-cosets. Set $J_3 = I_2 \setminus \{k_2,\ k_2'\}$. Now fix all S_i for all $i \in [1, r] \setminus J_3$, next assume that $|\sum_{i\in J_3} A_i|$ is maximal subject to all prior extremal assumptions still holding, and then fix $\sum_{i\in J_3} A_i$ up to translation. Repeating the above arguments, we again find that

$$|\sum_{i\in J_3} A_i| < \min\{|H_2|,\ \sum_{i\in J_3} |A_i| - |J_3| + 1\}.$$

Thus Property 1 implies that $H_3 := (\sum_{i\in J_2} A_i)$ has $5 \leq |H_3| < |H_2|$. Iterating the arguments of this paragraph, we obtain an infinite chain of subgroups $\infty > |G| > |H_1| > |H_2| > |H_3| > \ldots$, which is clearly impossible. This contradiction completes the proof. (Essentially, the only way the above process terminates after a finite number of steps is when we find enough elements from the same proper coset, whence Case 3 shows that the theorem holds for S.) $\square$

4. Distinct Solutions to a Linear Congruence

Let $r \in [2, n]$ and let $\alpha, a_1, \ldots, a_r \in \mathbb{Z}$. For each $x \in \mathbb{Z}$, we let $\overline{x} \in C_n$ denote x reduced modulo n. Consider the linear congruence

$$a_1x_1 + \ldots + a_rx_r \equiv \alpha \mod n.$$

Since the a_i are allowed to be zero, there is no loss of generality to assume $r = n$ when studying the above congruence, in which case we have

$$a_1x_1 + \ldots + a_nx_n \equiv \alpha \mod n. \tag{4.1}$$

It is a simple and well-known result that there is a solution $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ to (4.1) precisely when $\alpha \in \gcd(a_1, \ldots, a_n, n)\mathbb{Z}$. It is less immediate when a solution $(x_1, \ldots, x_n)$ with all x_i distinct modulo n exists. However, noting that the elements $a_1x_1 + \ldots + a_nx_n$ having the x_i distinct modulo n, when considered modulo n, are precisely the elements of $W \odot S$, where $W = 0(1)\cdot\ldots(n-1) \in \mathcal{F}(\mathbb{Z})$ and $S = \overline{a_1}\cdot\overline{a_2}\cdot\ldots\cdot\overline{a_n} \in \mathcal{F}(C_n)$, we then see that there existing a solution to (4.1) is equivalent to asking whether $\overline{\alpha} \in W \odot S$. If $n \geq 3$, then our main result Theorem 1.1 shows that $\overline{\alpha} \in W \odot S$ typically holds precisely when

$$\alpha \in \frac{(n-1)n}{2}a_1 + \gcd(a_2-a_1, a_3-a_1, \ldots, a_n-a_1, n)\mathbb{Z}, \tag{4.2}$$

the only exception being when, for some distinct j, k, $l \in [1,n]$, we have $a_j - a_l \equiv -a_k + a_l \mod n$, $\gcd(a_j - a_l, n) = 1$, and $a_i \equiv a_l \mod n$ for all $i \in [1,n] \setminus \{j,k\}$, in which case $\overline{\alpha} \in W \odot S$ instead holds precisely when

$$\alpha \in \frac{(n-1)n}{2}a_l + (\mathbb{Z} \setminus n\mathbb{Z}). \tag{4.3}$$

Thus Theorem 1.1 characterizes when a solution $(x_1,\ldots,x_n) \in \mathbb{Z}^n$ to (4.1) exists having all x_i distinct modulo n.

When $\alpha = 1$, the congruence (4.1) becomes

$$a_1x_1 + \ldots + a_nx_n \equiv 1 \mod n. \tag{4.4}$$

Fairly recently, in [1], solutions to (4.4) with all x_i distinct modulo n were constructed under the assumption that $\gcd(a_1,n) = \ldots = \gcd(a_k,n) = 1$ and $a_{k+1} = \ldots = a_n = 0$ for some $k < \varphi(n)$, where $\varphi(\cdot)$ denotes the Euler totient function. Additionally, [1, Theorem 2] proves the special case of Theorem 4.2 when n is prime, and Theorem 4.2 generalizes [1, Conjecture 3].

When $n = 2$, there are essentially only three possible choices for (a_1,a_2), namely $(0,0)$, $(0,1)$, and $(1,1)$. For $(0,0)$, there is no solution (x_1,x_2) to (4.4) with the x_i distinct modulo 2; for $(0,1)$, there is a solution (x_1,x_2) to (4.1) with the x_i distinct modulo 2 for all α; and for $(1,1)$, there is a solution (x_1,x_2) to (4.4) with the x_i distinct modulo 2 but no such solution to (4.1) for $\alpha = 0$. The following result gives some special instances of the characterization given by (4.2) and (4.3) for $n \geq 3$.

The first corollary addresses the question of when every $\alpha \in \mathbb{Z}$ has a solution $(x_1,\ldots,x_n)$ to (4.1) with the x_i distinct modulo n.

Corollary 4.1. *Let $n \geq 3$ and let $a_1,\ldots,a_n \in \mathbb{Z}$.*

1. *If, for some distinct j, k, $l \in [1,n]$, we have $a_j - a_l \equiv -a_k + a_l \mod n$, $\gcd(a_j - a_l, n) = 1$, and $a_i \equiv a_l \mod n$ for all $i \in [1,n] \setminus \{j,k\}$, then there is a solution $(x_1,\ldots,x_n) \in \mathbb{Z}^n$ to (4.4) with the x_i distinct modulo n but there is some $\alpha \neq 1$ for which there is no solution $(x_1,\ldots,x_n)$ to (4.1) with all the x_i distinct modulo n.*
2. *Otherwise, the following are equivalent.*
   (a) *For every $\alpha \in \mathbb{Z}$, there is a solution $(x_1,\ldots,x_n)$ to (4.1) with the x_i distinct modulo n.*
   (b) *For some $i \in [1,n]$, $\gcd(a_1 - a_i,\ldots,a_n - a_i, n) = 1$.*

Proof. Noting that $\gcd(a_1 - a_i,\ldots,a_n - a_i, n) = \gcd(a_1 - a_j,\ldots,a_n - a_j, n)$ for all i, $j \in [1,n]$, it follows that these are both simple consequences of (4.3) and (4.2). $\square$

The next result addresses the question of when (4.4) has a solution $(x_1,\ldots,x_n)$ with the x_i distinct modulo n. We remark that the arguments used below for $\alpha = 1$ would actually work for any $\alpha \in \mathbb{Z}$ with $\gcd(\alpha,n) = 1$.

Theorem 4.2. *Let $n \geq 2$ and let $a_1,\ldots,a_n \in \mathbb{Z}$.*

1. *If n is odd or some a_i is even, then (4.4) has a solution $(x_1,\ldots,x_n)$ with the x_i distinct modulo n if and only if $\gcd(a_2 - a_1, a_3 - a_1,\ldots,a_n - a_1, n) = 1$.*
2. *If $n \equiv 0 \mod 4$ and all a_i are odd, then (4.4) has no solution $(x_1,\ldots,x_n)$ with the x_i distinct modulo n.*
3. *If $n \equiv 2 \mod 4$ and all a_i are odd, then (4.4) has a solution $(x_1,\ldots,x_n)$ with the x_i distinct modulo n if and only if $\gcd(a_2 - a_1, a_3 - a_1,\ldots,a_n - a_1, n) = 2$.*

Proof. That the theorem holds for $n = 2$ can be easily checked, so we assume $n \geq 3$.

1. If the a_i satisfy the hypothesis of Corollary 4.1.1, then $\gcd(a_2 - a_1, a_3 - a_1,\ldots,a_n - a_1, n) = 1$ and Corollary 4.1.1 shows that (4.4) has a solution. Therefore assume the a_i do not satisfy the hypothesis of Corollary 4.1.1. If n is odd, then $\frac{(n-1)n}{2}a_1 \equiv 0 \mod n$, whence (4.2) shows that (4.4) has a solution $(x_1,\ldots,x_n)$ with the x_i distinct modulo n if and only if $\gcd(a_2 - a_1, a_3 - a_1,\ldots,a_n - a_1, n) = 1$. If some a_i is even, then we may w.l.o.g. re-index so that a_1 is even, whence $\frac{(n-1)n}{2}a_1 \equiv 0 \mod n$ again holds, completing the proof as before.

2. Since the a_i are odd and n is even, we have $2 \mid \gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n)$, in which case the hypotheses of Corollary 4.1.1 cannot hold for the a_i. Additionally, since $4 \mid n$, we have $2 \mid \frac{(n-1)n}{2} a_1$ as well, whence

$$\frac{(n-1)n}{2} a_1 + \gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n)\mathbb{Z} \subseteq 2\mathbb{Z}$$

and thus cannot contain 1. Hence (4.2) shows that (4.4) has no solution $(x_1, \ldots, x_n)$ with the x_i distinct modulo n.

3. As was the case in part 2, we have $2 \mid \gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n)$, so that the hypotheses of Corollary 4.1.1 cannot hold for the a_i. Since $n \equiv 2 \mod 4$ and a_1 is odd, we have $\frac{(n-1)n}{2} a_1 \equiv \frac{n}{2} \mod n$. Thus (4.2) shows that (4.4) has a solution $(x_1, \ldots, x_n)$ with the x_i distinct modulo n if and only if $\frac{n}{2} - 1 \in \gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n)\mathbb{Z}$. This condition rephrases as $\gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n) \mid (\frac{n}{2} - 1)$, which further rephrases as

$$\gcd\left(\frac{a_2 - a_1}{2}, \frac{a_3 - a_1}{2}, \ldots, \frac{a_n - a_1}{2}, \frac{n}{2}\right) \Bigm| \frac{n-2}{4}. \tag{4.5}$$

If there were a common factor $p \geq 2$ dividing both x and $\frac{x-1}{2}$, where $x \in \mathbb{Z}^+$, then $py = \frac{x-1}{2}$ and $pz = x$ for some positive integers y, $z \in \mathbb{Z}$, whence $2yp + 1 = x = pz$ follows, implying $p(z - 2y) = 1$, which contradicts that $p \geq 2$. Thus the integers x and $\frac{x-1}{2}$ can share no common factors. Applying this observation with $x = \frac{n}{2}$, we see that (4.5) holds precisely when $\gcd(\frac{a_2 - a_1}{2}, \frac{a_3 - a_1}{2}, \ldots, \frac{a_n - a_1}{2}, \frac{n}{2}) = 1$, which is equivalent to $\gcd(a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1, n) = 2$. This completes the final part of the theorem. $\square$

5. Consequences for Minimal Zero-Sum Sequences

We briefly recall the structure of minimal zero-sum sequences of maximal length in groups of rank 2. The following result was first shown as a conditional result in [48, Theorem 3.2], but by [16], [46], [9] and [19], the condition is satisfied.

Lemma 5.1 (cf. [48, Theorem 3.2]). *Let G be a finite abelian group of rank two, say $G \cong C_m \oplus C_{mn}$ with m, $n \in \mathbb{N}$ and $m \geq 2$. The minimal zero-sum sequences of maximal length are of the following forms.*

1. $S = e_j^{\operatorname{ord} e_j - 1} \prod_{i=1}^{\operatorname{ord} e_k} (x_i e_j + e_k)$, *where $\{e_1, e_2\}$ is a basis of G with* $\operatorname{ord} e_2 = mn$, $\{j, k\} = \{1, 2\}$, *and $x_i \in \mathbb{N}_0$ with* $\sum_{i=1}^{\operatorname{ord} e_k} x_i \equiv 1 \mod \operatorname{ord} e_j$.
2. $S = g_1^{sm-1} \prod_{i=1}^{(n+1-s)m} (x_i g_1 + g_2)$, *where $s \in [1, n]$, $\{g_1, g_2\}$ is a generating set of G with* $\operatorname{ord} g_2 = mn$ *and, in case $s \neq 1$, $mg_1 = mg_2$ and $x_i \in \mathbb{N}_0$ with* $\sum_{i=1}^{(n+1-s)m} x_i = m(n(n+1-s) - 1) + 1$.

In the second case of Lemma 5.1, the coefficients x_i are determined by equations only. But in the first case of Lemma 5.1, the coefficients x_i are determined by a congruence. Now suppose we are in case 1 and let G and S be as in Lemma 5.1. Then we may write S in the form

$$S = e_j^{\operatorname{ord} e_j - 1} \prod_{i=1}^{l} (x_i e_j + e_k)^{a_i},$$

where $\{e_1, e_2\}$ is a basis of G, $\operatorname{ord} e_1 = m$, $\operatorname{ord} e_2 = mn$, $\{j, k\} = \{1, 2\}$, $l \in [1, \operatorname{ord} e_j]$, $a_1, \ldots, a_l \in \mathbb{N}$ with $a_1 + \ldots + a_l = \operatorname{ord} e_k$, $x_1, \ldots, x_l \in [0, \operatorname{ord} e_j - 1]$ and all the x_i are distinct. Note $a_1 + \ldots + a_l = \operatorname{ord} e_k$ with each $a_i \geq 1$ implies that $l \leq \operatorname{ord} e_k$. Thus the characterization given by Lemma 5.1.1 easily implies that $|\operatorname{supp}(S)| \in [3, \min\{l, \operatorname{ord} e_j\}] = [3, m+1]$ (we cannot have $|\operatorname{supp}(S)| = 2$, as then all x_i from Lemma 5.1.1 would be equal modulo $\operatorname{ord} e_j$, in which case the congruence $x_1 + \ldots x_{\operatorname{ord} e_k} \equiv 1 \mod \operatorname{ord} e_j$ could not hold).

Here, we consider $\operatorname{ord} e_j - 1, a_1, \ldots, a_l$ as a multiplicity pattern of the elements arising in S. Thus two natural questions appear:

- Which multiplicity patterns can occur?
- How big can the support of S be?

We use the main result from Section 4 to answer these questions. In particular, we will show that any value of $[3, m+1]$ can be achieved for $|\operatorname{supp}(S)|$, apart from $m+1$ when $n = 1$ and $m \geq 3$, which, at least in the case $n = 1$, was originally shown in [22, Proposition 5.8.5]. First we set $a_i = 0$ for $i \in [l+1, \operatorname{ord} e_j]$, choose $x_{l+1}, \ldots, x_{\operatorname{ord} e_j} \in [0, \operatorname{ord} e_j - 1]$ such that all x_i are distinct, and obtain

$$a_1 x_1 + \ldots + a_{\operatorname{ord} e_j} x_{\operatorname{ord} e_j} \equiv 1 \mod \operatorname{ord} e_j \quad \text{and} \tag{5.1}$$

$$a_1 + \ldots + a_{\operatorname{ord} e_j} = \operatorname{ord} e_k. \tag{5.2}$$

Now there are three possible cases depending on $\operatorname{ord} e_j$ and $\operatorname{ord} e_k$.

Case 1. $\operatorname{ord} e_j = \operatorname{ord} e_k$, i.e. $n = 1$ and $\operatorname{ord} e_j = \operatorname{ord} e_k = m$. Then if equation (5.2) is satisfied, we must have either $a_1 = \ldots = a_{\operatorname{ord} e_j} = 1$ or $a_{\operatorname{ord} e_j} = 0$. Now we apply Theorem 4.2 and find that there is only a solution to (5.1) in the first case when $m = 2$, whence $|\operatorname{supp}(S)| = m + 1$ is only possible when $m = 2$, and that, in the second case, there is a solution to (5.1) for all choices of $a_1, \ldots, a_l$ with $\gcd(a_1, \ldots, a_l, \operatorname{ord} e_j) = 1 \mod n$, where $1 < l < \operatorname{ord} e_j = m$. In particular, taking the sequence $1^{l-1}(\operatorname{ord} e_j - l + 1)0^{\operatorname{ord} e_j - l}$ for $a_1 a_2 \cdot \ldots \cdot a_{\operatorname{ord} e_j}$, where $l \in [2, \operatorname{ord} e_j - 1]$, shows that any value of $|\operatorname{supp}(S)| \in [3, \operatorname{ord} e_j] = [3, m]$ is possible.

Case 2. $\operatorname{ord} e_k < \operatorname{ord} e_j$, i.e., $\operatorname{ord} e_k = m$ and $\operatorname{ord} e_j = mn \geq 4$ with $m, n \geq 2$. Then (5.2) forces $a_{\operatorname{ord} e_j} = 0$. Again we apply Theorem 4.2 and find that there is a solution to (5.1) for all choices of $a_1, \ldots, a_l$ with $\gcd(a_1, \ldots, a_l, \operatorname{ord} e_j) = 1 \mod n$, where $1 < l \leq \operatorname{ord} e_k < \operatorname{ord} e_j$. In particular, taking the sequence $1^{l-1}(\operatorname{ord} e_k - l + 1)0^{\operatorname{ord} e_j - l}$ for $a_1 a_2 \cdot \ldots \cdot a_{\operatorname{ord} e_j}$, where $l \in [2, \operatorname{ord} e_k] \subset [2, \operatorname{ord} e_j]$, shows that any value of $|\operatorname{supp}(S)| \in [3, \operatorname{ord} e_k + 1] = [3, m + 1]$ is possible.

Case 3. $\operatorname{ord} e_j < \operatorname{ord} e_k$, i.e., $\operatorname{ord} e_j = m$ and $\operatorname{ord} e_k = mn$. If $m = 2$, then (5.1) has a solution provided a_1 and a_2 are both odd. For $m \geq 3$, we apply Theorem 4.2 and obtain the following. The condition

$$\gcd(a_2 - a_1, a_3 - a_1 \ldots, a_{\operatorname{ord} e_j} - a_1, \operatorname{ord} e_j) \leq 2 \tag{5.3}$$

must always be fulfilled if (5.1) is to have a solution. Moreover, if m is odd or some a_i is even, then we must also have the inequality in (5.3) being strict, while if $4 \mid m$ and all a_i are odd, then no solution to (5.1) can be found. In particular, taking the sequence $1^{l-1}(\operatorname{ord} e_k - l + 1)0^{\operatorname{ord} e_j - l}$ for $a_1 a_2 \cdot \ldots \cdot a_{\operatorname{ord} e_j}$, where $l \in [2, \operatorname{ord} e_j - 1] = [1, m - 1]$, shows that any value of $|\operatorname{supp}(S)| \in [3, m]$ is possible. For $m \geq 3$, taking the sequence $1^{m-2}(2)(mn - m)$ for $a_1 a_2 \cdot \ldots \cdot a_{\operatorname{ord} e_j}$ shows that the value $|\operatorname{supp}(S)| = m + 1$ is also possible. Taking $(mn - 1)(1)$ for $a_1 a_2$ when $m = 2$ also shows that $|\operatorname{supp}(S)| = m + 1 = 3$ is possible when $m = 2$.

Note that, for groups of the form $G \cong C_m \oplus C_m$, all minimal zero-sum sequences of maximal length are of the form $S = e_1^{m-1} \prod_{i=1}^{m} (x_i e_1 + e_2)$, where $\{e_1, e_2\}$ is a basis of G with $\operatorname{ord} e_1 = \operatorname{ord} e_2 = m$ and $x_i \in \mathbb{N}_0$ with $\sum_{i=1}^{m} x_i \equiv 1 \mod m$. In this situation, only Case 1 appears.

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